ALGEBRAIC CONSTRUCTIONS IN THE CATEGORY OF LIE ALGEBROIDS

by
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Abstract

Using the notion of generalized Lie algebroid, we build the Lie algebroid generalized tangent bundle and we obtain a new point of view over (linear) connections theory.

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Contents

1 Introduction

Inspired by the general framework of Yang-Mills theory [2], presented synthetically in the following diagram:

$$\begin{tikzcd}
\left(E,\langle,\rangle_E\right) \arrow[d, "\pi"'] & \left(TM,[,]_{TM},(Id_{TM},Id_M),g\right) \arrow[d, "\tau_M"] \\
M \arrow[r, "Id_M"] & M
\end{tikzcd}$$

where:

1. (E,π,M) is a vector bundle,
2. $\langle,\rangle_E$ is an inner product for the module of sections $\Gamma\left(E,\pi,M\right)$,
3. $\left(\left(Id_{TM},Id_M\right),[,]_{TM}\right)$ is the usual Lie algebroid structure for the tangent vector bundle $\left(TM,\tau_M,M\right)$ and
4. $g\in\Gamma\left(\left(T^*M,\tau_M^*,M\right)\otimes\left(T^*M,\tau_M^*,M\right)\right)$ such that (M,g) is a Riemannian manifold,

we extend the notion of the Lie algebroid and we build the *Lie algebroid generalized tangent bundle.*

In particular, using the identity morphisms, we obtain a similar Lie algebroid with the *"prolongation Lie algebroid"* ([4-7], [10], [11]) and with the *"relativ tangent space"* [8].

The theory of (linear) connections constitutes undoubtedly one of most beautiful and most important chapter of differential geometry, which has been widely explored in the literature (see [3, 8, 9, 12, 13]).

In this paper, we introduce and develop a (linear) connections theory for fiber bundles, in general, and for vector bundles, in particular. Our main source of inspiration was provided by the papers [3], [8] and [9].

In this general framework, we can define the covariant derivatives of sections of a vector bundle (E, π, M) with respect to sections of a generalized Lie algebroid

$$\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right).$$

In particular, if we use the generalized Lie algebroid structure

$$\left([,]_{TM,Id_M}, (Id_{TM}, Id_M)\right)$$

for the tangent bundle (TM, τ_M, M) in our theory, then the linear connections obtained are similar with the classical linear connections for a vector bundle (E, π, M).

It is known that in Yang-Mills theory the set

$$Cov^0_{(E,\pi,M)}$$

of covariant derivatives for the vector bundle (E, π, M) such that

$$X\left(\langle u, v\rangle_E\right) = \langle D_X(u), v\rangle_E + \langle u, D_X(v)\rangle_E,$$

for any $X \in \mathcal{X}(M)$ and $u, v \in \Gamma(E, \pi, M)$, is very important, because the Yang-Mills theory is a variational theory which use (cf. [2]) the Yang-Mills functional

$$\begin{array}{ccc} Cov^0_{(E,\pi,M)} & \xrightarrow{\mathcal{YM}} & \mathbb{R} \\ D_X & \longmapsto & \frac{1}{2}\int\limits_M \left\|\mathbb{R}^{D_X}\right\|^2 v_g \end{array}$$

where $\mathbb{R}^{D_X}$ is the curvature.

Using this linear connections theory, we succeed to extend the set $Cov^0_{(E,\pi,M)}$ of Yang-Mills theory, because using all generalized Lie algebroid structures for the tangent bundle (TM, τ_M, M), we obtain all possible linear connections for the vector bundle (E, π, M).

Using our theory of linear connections we can obtain new and interesting results: formulas of Ricci type, identities of Bianchi and Cartan type, linear connection of Levi-Civita type,.....(see:[1])

2 Preliminaries

In general, if $\mathcal{C}$ is a category, then we denoted by $|\mathcal{C}|$ the class of objects and for any $A, B \in |\mathcal{C}|$, we denote by $\mathcal{C}(A, B)$ the set of morphisms of A source and B target.

Let **Vect**, **Liealg**, **Mod, Man**, **B** and $\mathbf{B^v}$ be the category of real vector spaces, Lie algebras, modules, manifolds, fiber bundles and vector bundles respectively.

We know that if $(E, \pi, M) \in |\mathbf{B^v}|$, then $(\Gamma(E, \pi, M), +, \cdot)$ is a $\mathcal{F}(M)$-module.

If $(\varphi, \varphi_0) \in \mathbf{B^v}((E, \pi, M), (E', \pi', M'))$ such that $\varphi_0 \in Diff(M, M')$, then we obtain the **Mod**-*morphism associated to the* $\mathbf{B^v}$-*morphism* (φ, φ_0)

$$\begin{array}{ccc} \Gamma(E, \pi, M) & \xrightarrow{\Gamma(\varphi, \varphi_0)} & \Gamma(E', \pi', M') \\ u & \longmapsto & \Gamma(\varphi, \varphi_0) u \end{array}$$

defined by

$$(\Gamma(\varphi, \varphi_0) u)(x') = \varphi\left(u_{\varphi_0^{-1}(x')}\right).$$

We know that if $(E, \pi, M) \in |\mathbf{B^v}|$ such that M is paracompact and if $A \subseteq M$ is closed, then for any section $u \in \Gamma\left(E_{|A}, i^*\pi, A\right)$ it exists $\tilde{u} \in \Gamma(E, \pi, M)$ such that $\tilde{u}_{|A} = u$.

Note: In the following, we consider only vector bundles with paracompact base.

Let $(\varphi, \varphi_0) \in \mathbf{B^v}((E, \pi, M), (E', \pi', M'))$ be. If, for every $y \in \varphi_0(M)$, we fixed $x_y \in M$ such that $\varphi_0(x_y) = y$, then we obtain an *extension* **Mod**-*morphism associated to the* $\mathbf{B^v}$-*morphism* (φ, φ_0)

$$\begin{array}{ccc} \Gamma(E, \pi, M) & \xrightarrow{\Gamma(\varphi, \varphi_0)} & \Gamma\left(E_{|\varphi_0(M)}, i^*\pi', \varphi_0(M)\right) \\ u & \longmapsto & \Gamma(\varphi, \varphi_0) u \end{array}$$

defined by

$$\Gamma(\varphi, \varphi_0) u(y) = \varphi\left(u_{x_y}\right).$$

As M' is paracompact, then it results that $\Gamma(\varphi, \varphi_0)$ can be regarded as **Mod**-morphism of $(\Gamma(E, \pi, M), +, \cdot)$ source and $(\Gamma(E', \pi', M') +, \cdot)$ target.

As any two extension **Mod**-morphisms associated to the $\mathbf{B^v}$-morphism (φ, φ_0) has the same properties, then an arbitrary extension **Mod**-morphism will be called *the extension* **Mod**-*morphism associated to the* $\mathbf{B^v}$-*morphism* (φ, φ_0).

We know that a Lie algebroid is a vector bundle $(F, \nu, N) \in |\mathbf{B^v}|$ such that there exists

$$(\rho, Id_N) \in \mathbf{B^v}((F, \nu, N), (TN, \tau_N, N))$$

and an operation

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[,]_F} & \Gamma(F, \nu, N) \\ (u, v) & \longmapsto & [u, v]_F \end{array}$$

with the following properties:

LA_1. the equality holds good

$$[u, f \cdot v]_F = f[u, v]_F + \Gamma(\rho, Id_N)(u) f \cdot v,$$

for all $u, v \in \Gamma(F, \nu, N)$ and $f \in \mathcal{F}(N)$,

LA_2. the 4-tuple $\left(\Gamma\left(F,\nu,N\right),+,\cdot,[,]_F\right)$ is a Lie $\mathcal{F}\left(N\right)$-algebra,

LA_3. the **Mod**-morphism $\Gamma\left(\rho,Id_N\right)$ is a **LieAlg**-morphism of $\left(\Gamma\left(F,\nu,N\right),+,\cdot,[,]_F\right)$ source and $\left(\Gamma\left(TN,\tau_N,N\right),+,\cdot,[,]_{TN}\right)$ target.

Obviously, in the definition of the Lie algebroid we use the **Mod**-morphism $\Gamma\left(\rho,Id_N\right)$ associated to the $\mathbf{B^v}$-morphism $\left(\rho,Id_N\right)$. Using the extension **Mod**-morphism associated to an arbitrary $\mathbf{B^v}$-morphism we can extend the notion of Lie algebroid and we obtain:

Definition 2.1 Let $M,N\in|\mathbf{Man}|$ and $h\in\mathbf{Man}\left(M,N\right)$ a surjective application. If $\left(F,\nu,N\right)\in|\mathbf{B^v}|$ such that there exists

$$\left(\rho,\eta\right)\in\mathbf{B^v}\left(\left(F,\nu,N\right),\left(TM,\tau_M,M\right)\right)$$

and an operation

$$\begin{array}{ccc}\Gamma\left(F,\nu,N\right)\times\Gamma\left(F,\nu,N\right) & \xrightarrow{[,]_{F,h}} & \Gamma\left(F,\nu,N\right)\\ \left(u,v\right) & \longmapsto & \left[u,v\right]_{F,h}\end{array}$$

with the following properties:

GLA_1. the equality holds good

$$\left[u,f\cdot v\right]_{F,h}=f\left[u,v\right]_{F,h}+\Gamma\left(Th\circ\rho,h\circ\eta\right)\left(u\right)f\cdot v,$$

for all $u,v\in\Gamma\left(F,\nu,N\right)$ and $f\in\mathcal{F}\left(N\right)$.

GLA_2. the 4-tuple $\left(\Gamma\left(F,\nu,N\right),+,\cdot,[,]_{F,h}\right)$ is a Lie $\mathcal{F}\left(N\right)$-algebra,

GLA_3. the **Mod**-morphism $\Gamma\left(Th\circ\rho,h\circ\eta\right)$ is a **LieAlg**-morphism of $\left(\Gamma\left(F,\nu,N\right),+,\cdot,[,]_{F,h}\right)$ source and $\left(\Gamma\left(TN,\tau_N,N\right),+,\cdot,[,]_{TN}\right)$ target, then we will say that *the triple*

$$\left(\left(F,\nu,N\right),[,]_{F,h},\left(\rho,\eta\right)\right)\tag{2.1}$$

is a generalized Lie algebroid. The couple $\left([,]_{F,h},\left(\rho,\eta\right)\right)$ will be called *generalized Lie algebroid structure.*

Let $\left(\left(F,\nu,N\right),[,]_{F,h},\left(\rho,\eta\right)\right)$ be a generalized Lie algebroid.

- Locally, for any $\alpha,\beta\in\overline{1,p}$, we set $\left[t_\alpha,t_\beta\right]_F\overset{put}{=}L_{\alpha\beta}^\gamma t_\gamma$. We easily obtain that $L_{\alpha\beta}^\gamma=-L_{\beta\alpha}^\gamma$, for any $\alpha,\beta,\gamma\in\overline{1,p}$.

The real local functions $L_{\alpha\beta}^\gamma,\ \alpha,\beta,\gamma\in\overline{1,p}$ will be called the *structure functions of the generalized Lie algebroid* $\left(\left(F,\nu,N\right),[,]_{F,h},\left(\rho,\eta\right)\right)$.

- We assume that $\left(F,\nu,N\right)$ is a vector bundle with type fibre the real vector space $\left(\mathbb{R}^p,+,\cdot\right)$ and structure group a Lie subgroup of $\left(\mathbf{GL}\left(p,\mathbb{R}\right),\cdot\right)$.

We take $\left(x^{i},y^{i}\right)$ as canonical local coordinates on $\left(TM,\tau_{M},M\right)$, where $i\in\overline{1,m}$. Consider

$$\left(x^{i},y^{i}\right)\longrightarrow\left(x^{\acute{i}}\left(x^{i}\right),y^{\acute{i}}\left(x^{i},y^{i}\right)\right)$$

a change of coordinates on (TM,τ_{M},M). Then the coordinates y^{i} change to $y^{\acute{i}}$ by the rule:

$$y^{\acute{i}}=\tfrac{\partial x^{\acute{i}}}{\partial x^{i}}y^{i}. \tag{2.2}$$

We take $(\varkappa^{\tilde{i}},z^{\alpha})$ as canonical local coordinates on (F,ν,N), where $\tilde{i}\in\overline{1,n}$, $\alpha\in\overline{1,p}$. Consider

$$\left(\varkappa^{\tilde{i}},z^{\alpha}\right)\longrightarrow\left(\varkappa^{\tilde{i}},z^{\acute{\alpha}}\right)$$

a change of coordinates on (F,ν,N). Then the coordinates z^{α} change to $z^{\acute{\alpha}}$ by the rule:

$$z^{\acute{\alpha}}=\Lambda_{\alpha}^{\acute{\alpha}}z^{\alpha}. \tag{2.3}$$

- We assume that $(\theta,\mu)\overset{put}{=}(Th\circ\rho,h\circ\eta)$. If $z^{\alpha}t_{\alpha}\in\Gamma\left(F,\nu,N\right)$ is arbitrary, then

$$\begin{array}{c}\Gamma\left(Th\circ\rho,h\circ\eta\right)\left(z^{\alpha}t_{\alpha}\right)f\left(h\circ\eta\left(\varkappa\right)\right)=\\=\left(\theta_{\alpha}^{\tilde{i}}z^{\alpha}\frac{\partial f}{\partial\varkappa^{\tilde{i}}}\right)\left(h\circ\eta\left(\varkappa\right)\right)=\left(\left(\rho_{\alpha}^{i}\circ h\right)\left(z^{\alpha}\circ h\right)\frac{\partial f\circ h}{\partial x^{i}}\right)\left(\eta\left(\varkappa\right)\right),\end{array} \tag{2.4}$$

for any $f\in\mathcal{F}\left(N\right)$ and $\varkappa\in N$.

The coefficients ρ_{α}^{i} respectively $\theta_{\alpha}^{\tilde{i}}$ change to $\rho_{\acute{\alpha}}^{\acute{i}}$ respectively $\theta_{\acute{\alpha}}^{\tilde{i}}$ by the rule:

$$\rho_{\acute{\alpha}}^{\acute{i}}=\Lambda_{\acute{\alpha}}^{\alpha}\rho_{\alpha}^{i}\frac{\partial x^{\acute{i}}}{\partial x^{i}}, \tag{2.5}$$

respectively

$$\theta_{\acute{\alpha}}^{\tilde{i}}=\Lambda_{\acute{\alpha}}^{\alpha}\theta_{\alpha}^{\tilde{i}}\frac{\partial\varkappa^{\tilde{i}}}{\partial\varkappa^{\tilde{i}}}, \tag{2.6}$$

where

$$\left\|\Lambda_{\acute{\alpha}}^{\alpha}\right\|=\left\|\Lambda_{\alpha}^{\acute{\alpha}}\right\|^{-1}.$$

Remark 2.2 The following equalities hold good:

$$\rho_{\alpha}^{i}\circ h\frac{\partial f\circ h}{\partial x^{i}}=\left(\theta_{\alpha}^{\tilde{i}}\frac{\partial f}{\partial\varkappa^{\tilde{i}}}\right)\circ h,\forall f\in\mathcal{F}\left(N\right). \tag{2.7}$$

and

$$\left(L_{\alpha\beta}^{\gamma}\circ h\right)\left(\rho_{\gamma}^{k}\circ h\right)=\left(\rho_{\alpha}^{i}\circ h\right)\frac{\partial\left(\rho_{\beta}^{k}\circ h\right)}{\partial x^{i}}-\left(\rho_{\beta}^{j}\circ h\right)\frac{\partial\left(\rho_{\alpha}^{k}\circ h\right)}{\partial x^{j}}. \tag{2.8}$$

Theorem 2.1 *Let $M\in\left|\mathbf{Man}_{m}\right|$ and $g,h\in Iso_{\mathbf{Man}}\left(M\right)$ be. Using the tangent $\mathbf{B^{v}}$-morphism (Tg,g) and the operation*

$$\begin{array}{ccc}\Gamma\left(TM,\tau_{M},M\right)\times\Gamma\left(TM,\tau_{M},M\right) & \xrightarrow{[,]_{TM,h}} & \Gamma\left(TM,\tau_{M},M\right)\\ \left(u,v\right) & \longmapsto & \left[u,v\right]_{TM,h}\end{array}$$

where

$$[u,v]_{TM,h}=\Gamma\left(T\left(h\circ g\right)^{-1},\left(h\circ g\right)^{-1}\right)\left(\left[\Gamma\left(T\left(h\circ g\right),h\circ g\right)u,\Gamma\left(T\left(h\circ g\right),h\circ g\right)v\right]_{TM}\right),$$

for any $u,v\in\Gamma\left(TM,\tau_M,M\right)$, *we obtain that*

$$\left(\left(TM,\tau_M,M\right),\left(Tg,g\right),[,]_{TM,h}\right)$$

is a generalized Lie algebroid.

For any **Man**-isomorphisms g and h we obtain new and interesting generalized Lie algebroid structures for the tangent vector bundle $\left(TM,\tau_M,M\right)$. For any base $\left\{t_\alpha,\ \alpha\in\overline{1,m}\right\}$ of the module of sections $\left(\Gamma\left(TM,\tau_M,M\right),+,\cdot\right)$ we obtain the structure functions

$$L_{\alpha\beta}^{\gamma}=\left(\theta_\alpha^i\frac{\partial\theta_\beta^j}{\partial x^i}-\theta_\beta^i\frac{\partial\theta_\alpha^j}{\partial x^i}\right)\tilde{\theta}_j^\gamma,\ \alpha,\beta,\gamma\in\overline{1,m}$$

where

$$\theta_\alpha^i,\ i,\alpha\in\overline{1,m}$$

are real local functions such that

$$\Gamma\left(T\left(h\circ g\right),h\circ g\right)\left(t_\alpha\right)=\theta_\alpha^i\frac{\partial}{\partial x^i}$$

and

$$\tilde{\theta}_j^\gamma,\ i,\gamma\in\overline{1,m}$$

are real local functions such that

$$\Gamma\left(T\left(h\circ g\right)^{-1},\left(h\circ g\right)^{-1}\right)\left(\frac{\partial}{\partial x^j}\right)=\tilde{\theta}_j^\gamma t_\gamma.$$

In particular, using arbitrary basis for the module of sections and arbitrary isometries (symmetries, translations, rotations,...) for the Euclidean 3-dimensional space Σ, we obtain a lot of generalized Lie algebroid structures for the tangent vector bundle $(T\Sigma,\tau_\Sigma,\Sigma)$.

We assume that $\left(\left(F,\nu,N\right),[,]_F,\left(\rho,Id_N\right)\right)$ is a Lie algebroid and let $h\in\mathbf{Man}\left(N,N\right)$ be a surjective application. Let $\mathcal{AF}_F$ be a vector fibred $(n+p)$-atlas for the vector bundle (F,ν,N) and let $\mathcal{AF}_{TN}$ be a vector fibred $(n+n)$-atlas for the vector bundle (TN,τ_N,N).

If $(U,\xi_U)\in\mathcal{AF}_{TN}$ and $(V,s_V)\in\mathcal{AF}_F$ such that $U\cap h^{-1}\left(V\right)\neq\phi$, then we define the application

$$\begin{array}{ccc}\tau_N^{-1}(U\cap h^{-1}(V))) & \xrightarrow{\bar{\xi}_{U\cap h^{-1}(V)}} & \left(U\cap h^{-1}(V)\right)\times\mathbb{R}^n\\ \left(\varkappa,u\left(\varkappa\right)\right) & \longmapsto & \left(\varkappa,\xi_{U,\varkappa}^{-1}u\left(\varkappa\right)\right).\end{array}$$

Proposition 2.1 *The set*

$$\overline{\mathcal{AF}}_{TN}\overset{put}{=}\bigcup_{\substack{(U,\xi_U)\in\mathcal{AF}_{TN},\ (V,s_V)\in\mathcal{AF}_F\\ U\cap h^{-1}(V)\neq\phi}}\left\{\left(U\cap h^{-1}\left(V\right),\bar{\xi}_{U\cap h^{-1}(V)}\right)\right\}$$

is a vector fibred $n+n$*-atlas for the vector bundle* $\left(TN,\tau_N,N\right)$.

If $X=X^{\tilde{\imath}}\frac{\partial}{\partial \varkappa^{\tilde{\imath}}}\in\Gamma\left(TN,\tau_N,N\right)$, *then we obtain the section*

$$\bar{X}=\bar{X}^{\tilde{\imath}}\circ h\frac{\partial}{\partial \bar{\varkappa}^{\tilde{\imath}}}\in\Gamma\left(TN,\tau_N,N\right),$$

such that $\bar{X}\left(\bar{\varkappa}\right)=X\left(h\left(\bar{\varkappa}\right)\right)$, *for any* $\bar{\varkappa}\in U\cap h^{-1}\left(V\right)$.

The set $\left\{\frac{\partial}{\partial \bar{\varkappa}^{\tilde{\imath}}},\ \tilde{\imath}\in\overline{1,n}\right\}$ is a base for the $\mathcal{F}\left(N\right)$-module $\left(\Gamma\left(TN,\tau_N,N\right),+,\cdot\right)$.

Theorem 2.2 *If we consider the operation*

$$\Gamma\left(F,\nu,N\right)\times\Gamma\left(F,\nu,N\right)\ \xrightarrow{[,]_{F,h}}\ \Gamma\left(F,\nu,N\right)$$

defined by

$$\begin{array}{ll}\left[t_\alpha,t_\beta\right]_{F,h} & =\left(L_{\alpha\beta}^{\gamma}\circ h\right)t_\gamma,\\ \left[t_\alpha,ft_\beta\right]_{F,h} & =f\left(L_{\alpha\beta}^{\gamma}\circ h\right)t_\gamma+\rho_\alpha^{\tilde{\imath}}\circ h\dfrac{\partial f}{\partial \bar{\varkappa}^{\tilde{\imath}}}t_\beta,\\ \left[ft_\alpha,t_\beta\right]_{F,h} & =-\left[t_\beta,ft_\alpha\right]_{F,h},\end{array}$$

for any $f\in\mathcal{F}\left(N\right)$, *then* $\Big(\left(F,\nu,N\right),[,]_{F,h},\left(\rho,Id_N\right)\Big)$ *is a generalized Lie algebroid.*

This generalized Lie algebroid is called *the generalized Lie algebroid associated to the Lie algebroid* $\left(\left(F,\nu,N\right),[,]_F,\left(\rho,Id_N\right)\right)$ *and to the surjective application* $h\in\mathbf{Man}\left(N,N\right)$.

In particular, if $h=Id_N$, then the generalized Lie algebroid

$$\Big(\left(F,\nu,N\right),[,]_{F,Id_N},\left(\rho,Id_N\right)\Big)$$

will be called *the generalized Lie algebroid associated to the Lie algebroid*

$$\left(\left(F,\nu,N\right),[,]_F,\left(\rho,Id_N\right)\right).$$

Let $\mathcal{AF}_{TM}$ be a vector fibred $(m+m)$-atlas for the vector bundle (TM,τ_M,M) and let $(h^*F,h^*\nu,M)$ be the pull-back vector bundle through h. If $(U,\xi_U)\in\mathcal{AF}_{TM}$ and $(V,s_V)\in\mathcal{AF}_F$ such that $U\cap h^{-1}\left(V\right)\neq\phi$, then we define the application

$$\begin{array}{ccc}h^*\nu^{-1}(U\cap h^{-1}(V))) & \xrightarrow{\bar{s}_{U\cap h^{-1}(V)}} & \left(U\cap h^{-1}(V)\right)\times\mathbb{R}^p\\ \left(\varkappa,z\left(h\left(\varkappa\right)\right)\right) & \longmapsto & \left(\varkappa,t_{V,h(\varkappa)}^{-1}z\left(h\left(\varkappa\right)\right)\right).\end{array}$$

Proposition 2.2 *The set*

$$\overline{\mathcal{AF}}_F\overset{put}{=}\bigcup_{\substack{(U,\xi_U)\in\mathcal{AF}_{TM},\ (V,s_V)\in\mathcal{AF}_F\\ U\cap h^{-1}(V)\neq\phi}}\left\{\left(U\cap h^{-1}\left(V\right),\bar{s}_{U\cap h^{-1}(V)}\right)\right\}$$

is a vector fibred $m+p$*-atlas for the vector bundle* $\left(h^*F,h^*\nu,M\right)$.

If

$$z=z^\alpha t_\alpha\in\Gamma\left(F,\nu,N\right),$$

then we obtain the section

$$Z=\left(z^\alpha\circ h\right)T_\alpha\in\Gamma\left(h^*F,h^*\nu,M\right)$$

such that

$$Z\left(x\right)=z\left(h\left(x\right)\right),$$

for any $x \in U \cap h^{-1}(V)$.

In addition, we obtain the $\mathbf{B^v}$-morphism

$$\begin{array}{ccc} h^*F & \hookrightarrow & F \\ h^*\nu \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array} \tag{2.9}$$

Theorem 2.3 *Let* $\left(\overset{h^*F}{\rho}, Id_M\right)$ *be the* $\mathbf{B^v}$*-morphism of* $(h^*F, h^*\nu, M)$ *source and* (TM, τ_M, M) *target, where*

$$\begin{array}{rcl} h^*F & \overset{h^*F}{\underset{\longrightarrow}{\rho}} & TM \\ Z^\alpha T_\alpha(x) & \longmapsto & \left(Z^\alpha \cdot \rho_\alpha^i \circ h\right) \dfrac{\partial}{\partial x^i}(x) \end{array} \tag{2.10}$$

Using the operation

$$\Gamma(h^*F, h^*\nu, M) \times \Gamma(h^*F, h^*\nu, M) \xrightarrow{[,]_{h^*F}} \Gamma(h^*F, h^*\nu, M)$$

defined by

$$\begin{array}{ll} [T_\alpha, T_\beta]_{h^*F} & = \left(L_{\alpha\beta}^\gamma \circ h\right) T_\gamma, \\ [T_\alpha, fT_\beta]_{h^*F} & = f\left(L_{\alpha\beta}^\gamma \circ h\right) T_\gamma + \left(\rho_\alpha^i \circ h\right) \dfrac{\partial f}{\partial x^i} T_\beta, \\ [fT_\alpha, T_\beta]_{h^*F} & = -[T_\beta, fT_\alpha]_{h^*F}, \end{array} \tag{2.11}$$

for any $f \in \mathcal{F}(M)$, *it results that*

$$\left((h^*F, h^*\nu, M), [,]_{h^*F}, \left(\overset{h^*F}{\rho}, Id_M\right)\right)$$

is a Lie algebroid which is called the pull-back Lie algebroid of the generalized Lie algebroid $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$.

3 The Lie algebroid generalized tangent bundle

We consider the following diagram:

$$\begin{array}{ccc} E & & \left(F, [,]_{F,h}, (\rho, \eta)\right) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array} \tag{3.1}$$

where (E, π, M) is a fiber bundle and $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$ is a generalized Lie algebroid.

We take (x^i, y^a) as canonical local coordinates on (E, π, M), where $i \in \overline{1, m}$ and $a \in \overline{1, r}$. Let

$$(x^i, y^a) \longrightarrow \left(x^{i'}(x^i), y^{a'}(x^i, y^a)\right)$$

be a change of coordinates on (E,π,M). Then the coordinates y^{a} change to $y^{\acute{a}}$ by the rule:

$$y^{\acute{a}}=\frac{\partial y^{\acute{a}}}{\partial y^{a}}y^{a}. \tag{3.2}$$

In particular, if (E,π,M) is vector bundle, then the coordinates y^{a} change to $y^{\acute{a}}$ by the rule:

$$y^{\acute{a}}=M_{a}^{\acute{a}}y^{a}. \tag{3.2$'$}$$

Let

$$\left(\pi^{\ast}\left(h^{\ast}F\right),\pi^{\ast}\left(h^{\ast}\nu\right),E\right),\left[,\right]_{\pi^{\ast}\left(h^{\ast}F\right)},\left(\overset{\pi^{\ast}\left(h^{\ast}F\right)}{\rho},Id_{E}\right)$$

be the pull-back Lie algebroid of the Lie algebroid

$$\left(h^{\ast}F,h^{\ast}\nu,M\right),\left[,\right]_{h^{\ast}F},\left(\overset{h^{\ast}F}{\rho},Id_{M}\right).$$

If $z=z^{\alpha}t_{\alpha}\in\Gamma\left(F,\nu,N\right)$, then, using the vector fibred $(m+r)+p$-atlas $\widetilde{\mathcal{AF}}_{\pi^{\ast}\left(h^{\ast}F\right)}$, we obtain the section

$$\tilde{Z}=\left(z^{\alpha}\circ h\circ\pi\right)\tilde{T}_{\alpha}\in\Gamma\left(\pi^{\ast}\left(h^{\ast}F\right),\pi^{\ast}\left(h^{\ast}\nu\right),E\right)$$

such that $\tilde{Z}\left(u_{x}\right)=z\left(h\left(x\right)\right)$, for any $u_{x}\in\pi^{-1}\left(U\cap h^{-1}V\right)$.

For any sections

$$\tilde{Z}^{\alpha}\tilde{T}_{\alpha}\in\Gamma\left(\pi^{\ast}\left(h^{\ast}F\right),\pi^{\ast}\left(h^{\ast}F\right),E\right)$$

and

$$Y^{a}\frac{\partial}{\partial y^{a}}\in\Gamma\left(VTE,\tau_{E},E\right)$$

we obtain the section

$$\begin{array}{l}\tilde{Z}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y^{a}\frac{\partial}{\partial\tilde{y}^{a}}=:\tilde{Z}^{\alpha}\left(\tilde{T}_{\alpha}\oplus\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\frac{\partial}{\partial x^{i}}\right)+Y^{a}\left(0_{\pi^{\ast}\left(h^{\ast}F\right)}\oplus\frac{\partial}{\partial y^{a}}\right)\\ =\tilde{Z}^{\alpha}\tilde{T}_{\alpha}\oplus\left(\tilde{Z}^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\frac{\partial}{\partial x^{i}}+Y^{a}\frac{\partial}{\partial y^{a}}\right)\in\Gamma\left(\pi^{\ast}\left(h^{\ast}F\right)\oplus TE,\overset{\oplus}{\pi},E\right).\end{array}$$

Since we have

$$\begin{array}{c}\tilde{Z}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y^{a}\frac{\partial}{\partial\tilde{y}^{a}}=0\\ \Updownarrow\\ \tilde{Z}^{\alpha}\tilde{T}_{\alpha}=0\ \wedge\tilde{Z}^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\frac{\partial}{\partial x^{i}}+Y^{a}\frac{\partial}{\partial y^{a}}=0,\end{array}$$

it implies $\tilde{Z}^{\alpha}=0,\ \alpha\in\overline{1,p}$ and $Y^{a}=0,\ a\in\overline{1,r}$.

Therefore the sections $\frac{\partial}{\partial\tilde{z}^{1}},...,\frac{\partial}{\partial\tilde{z}^{p}},\frac{\partial}{\partial\tilde{y}^{1}},...,\frac{\partial}{\partial\tilde{y}^{r}}$ are linearly independent.

We consider the vector subbundle $\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)$ of the vector bundle $\left(\pi^{\ast}\left(h^{\ast}F\right)\oplus TE,\overset{\oplus}{\pi},E\right)$, for which the $\mathcal{F}\left(E\right)$-module of sections is the $\mathcal{F}\left(E\right)$-submodule of $\left(\Gamma\left(\pi^{\ast}\left(h^{\ast}F\right)\oplus TE,\overset{\oplus}{\pi},E\right),+,\cdot\right)$, generated by the set of sections $\left(\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{y}^{a}}\right)$.

The base sections

$$\left(\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{y}^{a}}\right)\overset{put}{=}\left(\tilde{\partial}_{\alpha},\dot{\tilde{\partial}}_{a}\right) \tag{3.4}$$

will be called the *natural* (ρ,η)*-base.*

The matrix of coordinate transformation on $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ at a change of fibred charts is

$$\left\|\begin{array}{cc}\Lambda_{\alpha}^{\acute{\alpha}}\circ h\circ\pi & 0\\ \left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\dfrac{\partial y^{\acute{a}}}{\partial x^{i}} & \dfrac{\partial y^{\acute{a}}}{\partial y^{a}}\end{array}\right\|. \tag{3.5}$$

In particular, if (E,π,M) is a vector bundle, then the matrix of coordinate transformation on $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ at a change of fibred charts is

$$\left\|\begin{array}{cc}\Lambda_{\alpha}^{\acute{\alpha}}\circ h\circ\pi & 0\\ \left(\rho_{a}^{i}\circ h\circ\pi\right)\dfrac{\partial M_{b}^{\acute{a}}\circ\pi}{\partial x_{i}}y^{b} & M_{a}^{\acute{a}}\circ\pi\end{array}\right\|. \tag{3.6}$$

Easily we obtain

Theorem 3.1 *Let* $(\tilde{\rho},Id_E)$ *be the* $\mathbf{B^v}$*-morphism of* $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ *source and* (TE,τ_E,E) *target, where*

$$\begin{array}{c}(\rho,\eta)\,TE\xrightarrow{\tilde{\rho}}TE\\ \left(\tilde{Z}^{\alpha}\dfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y^{a}\dfrac{\partial}{\partial\tilde{y}^{a}}\right)(u_x)\longmapsto\left(\tilde{Z}^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\dfrac{\partial}{\partial x^{i}}+Y^{a}\dfrac{\partial}{\partial y^{a}}\right)(u_x)\end{array} \tag{3.7}$$

Using the operation

$$\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)^2\xrightarrow{[,]_{(\rho,\eta)TE}}\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$$

defined by

$$\begin{array}{l}\left[\left(\tilde{Z}_1^{\alpha}\dfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y_1^{a}\dfrac{\partial}{\partial\tilde{y}^{a}}\right),\left(\tilde{Z}_2^{\beta}\dfrac{\partial}{\partial\tilde{z}^{\beta}}+Y_2^{b}\dfrac{\partial}{\partial\tilde{y}^{b}}\right)\right]_{(\rho,\eta)TE}\\ =\left[\tilde{Z}_1^{\alpha}\tilde{T}_a,\tilde{Z}_2^{\beta}\tilde{T}_{\beta}\right]_{\pi^*(h^*F)}\oplus\left[\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\tilde{Z}_1^{\alpha}\dfrac{\partial}{\partial x^{i}}+Y_1^{a}\dfrac{\partial}{\partial y^{a}},\right.\\ \qquad\qquad\left.\left(\rho_{\beta}^{i}\circ h\circ\pi\right)\tilde{Z}_2^{\beta}\dfrac{\partial}{\partial x^{i}}+Y_2^{b}\dfrac{\partial}{\partial y^{b}}\right]_{TE},\end{array} \tag{3.8}$$

for any $\left(\tilde{Z}_1^{\alpha}\dfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y_1^{a}\dfrac{\partial}{\partial\tilde{y}^{a}}\right)$ *and* $\left(\tilde{Z}_2^{\beta}\dfrac{\partial}{\partial\tilde{z}^{\beta}}+Y_2^{b}\dfrac{\partial}{\partial\tilde{y}^{b}}\right)$, *we obtain that the couple*

$$\left([,]_{(\rho,\eta)TE},(\tilde{\rho},Id_E)\right)$$

is a Lie algebroid structure for the vector bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$.

Remark 3.2 In particular, if $h=Id_M$ and $[,]_{TM}$ is the usual Lie bracket, it results that the Lie algebroid

$$\left(((Id_{TM},Id_M)\,TE,(Id_{TM},Id_M)\,\tau_E,E)\,,[,]_{(Id_{TM},Id_M)TE},\left(\widetilde{Id_{TM}},Id_E\right)\right)$$

is isomorphic with the usual Lie algebroid

$$\left(\left(TE,\tau_E,E\right),\left[,\right]_{TE},\left(Id_{TE},Id_E\right)\right).$$

This is a reason for which the Lie algebroid

$$\left(\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right),\left[,\right]_{\left(\rho,\eta\right)TE},\left(\tilde{\rho},Id_E\right)\right)\ ,$$

will be called the *Lie algebroid generalized tangent bundle.*

The vector bundle $\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$ will be called the *generalized tangent bundle.*

4 (Linear) (ρ,η)-connections

We consider the diagram:

$$\begin{array}{ccc} E & & \left(F,\left[,\right]_{F,h},\left(\rho,\eta\right)\right) \\ \downarrow \pi & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where $(E,\pi,M)\in|\mathbf{B}|$ and $\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$ is a generalized Lie algebroid.

Let

$$\left(\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right),\left[,\right]_{\left(\rho,\eta\right)TE},\left(\tilde{\rho},Id_E\right)\right)$$

be the Lie algebroid generalized tangent bundle of fiber bundle (E,π,M).

We consider the $\mathbf{B^v}$-morphism $\left(\left(\rho,\eta\right)\pi!,Id_E\right)$ given by the commutative diagram

$$\begin{array}{ccc} \left(\rho,\eta\right)TE & \xrightarrow{\left(\rho,\eta\right)\pi!} & \pi^*\left(h^*F\right) \\ {\scriptstyle \left(\rho,\eta\right)\tau_E}\downarrow & & \downarrow {\scriptstyle pr_1} \\ E & \xrightarrow{id_E} & E \end{array} \tag{4.1}$$

This is defined as:

$$\left(\rho,\eta\right)\pi!\left(\left(\tilde{Z}^\alpha\frac{\partial}{\partial\tilde{z}^\alpha}+Y^a\frac{\partial}{\partial\tilde{y}^a}\right)(u_x)\right)=\left(\tilde{Z}^\alpha\tilde{T}_\alpha\right)(u_x)\,, \tag{4.2}$$

for any $\left(\tilde{Z}^\alpha\dfrac{\partial}{\partial\tilde{z}^\alpha}+Y^a\dfrac{\partial}{\partial\tilde{y}^a}\right)\in\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right).$

Using the $\mathbf{B^v}$-morphism $\left(\left(\rho,\eta\right)\pi!,Id_E\right)$, and the the $\mathbf{B^v}$-morphism (2.9) we obtain the *tangent* (ρ,η)*-application* $\left(\left(\rho,\eta\right)T\pi,h\circ\pi\right)$ of $\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$ source and (F,ν,N) target.

Definition 4.1 The kernel of the tangent (ρ,η)-application is written $\left(V\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$ and it is called *the vertical subbundle.*

We remark that the set $\left\{\dfrac{\partial}{\partial\tilde{y}^a},\ a\in\overline{1,r}\right\}$ is a base for the $\mathcal{F}(E)$-module

$$\left(\Gamma\left(V\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right),+,\cdot\right).$$

Proposition 4.1 *The short sequence of vector bundles*

$$
(4.3)\qquad
\begin{array}{ccccccccc}
0 & \overset{i}{\hookrightarrow} & V(\rho,\eta)TE & \overset{i}{\hookrightarrow} & (\rho,\eta)TE & \xrightarrow{(\rho,\eta)\pi!} & \pi^*\left(h^*F\right) & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E
\end{array}
$$

is exact.

Definition 4.2 *A* **Man**-morphism $(\rho,\eta)\Gamma$ of $(\rho,\eta)TE$ source and $V(\rho,\eta)TE$ target defined by

$$
(4.4)\qquad (\rho,\eta)\Gamma\left(\tilde{Z}^\alpha\frac{\partial}{\partial\tilde{z}^\alpha}+Y^a\frac{\partial}{\partial\tilde{y}^a}\right)(u_x)=\left(Y^a+(\rho,\eta)\Gamma^a_\alpha\tilde{Z}^\alpha\right)\frac{\partial}{\partial\tilde{y}^a}(u_x),
$$

such that the $\mathbf{B^v}$-morphism $((\rho,\eta)\Gamma, Id_E)$ is a split to the left in the previous exact sequence, will be called (ρ,η)*-connection for the fiber bundle* (E,π,M).

The (ρ, Id_M)-connection will be called ρ*-connection* and will be denoted $\rho\Gamma$ and the (Id_{TM}, Id_M)-connection will be called *connection* and will be denoted Γ.

Definition 4.3 If $(\rho,\eta)\Gamma$ is a (ρ,η)-connection for the fiber bundle (E,π,M), then the kernel of the $\mathbf{B^v}$-morphism $((\rho,\eta)\Gamma, Id_E)$ is written $(H(\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ and will be called the *horizontal vector subbundle.*

Definition 4.4 If $(E,\pi,M)\in|\mathbf{B}|$, then the **B**-morphism (Π,π) defined by the commutative diagram

$$
(4.5)\qquad
\begin{array}{ccc}
V(\rho,\eta)TE & \xrightarrow{\Pi} & E\\
{\scriptstyle (\rho,\eta)\tau_E}\downarrow & & \downarrow{\scriptstyle \pi}\\
E & \xrightarrow{\pi} & M
\end{array}
$$

such that the components of the image of the vector $Y^a\frac{\partial}{\partial\tilde{y}^a}(u_x)$ are the real numbers $Y^1(u_x),...,Y^r(u_x)$ will be called the *canonical projection* $\mathbf{B}$*-morphism.*

In particular, if $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left\{s_a, a\in\overline{1,r}\right\}$ is the base of $\mathcal{F}(M)$-module of sections $(\Gamma(E,\pi,M),+,\cdot)$, then Π is defined by

$$
(4.6)\qquad \Pi\left(Y^a\frac{\partial}{\partial\tilde{y}^a}(u_x)\right)=Y^a(u_x)\,s_a(x).
$$

Theorem 4.1 *If* $(\rho,\eta)\Gamma$ *is a* (ρ,η)*-connection for the fiber bundle* (E,π,M), *then its components satisfy the law of transformation*

$$
(4.7)\qquad (\rho,\eta)\Gamma^{a'}_{\gamma'}=\frac{\partial y^{a'}}{\partial y^a}\left[\rho^i_\gamma\circ(h\circ\pi)\frac{\partial y^a}{\partial x^i}+(\rho,\eta)\Gamma^a_\gamma\right]\Lambda^\gamma_{\gamma'}\circ(h\circ\pi).
$$

If $(\rho,\eta)\Gamma$ *is a* (ρ,η)*-connection for the vector bundle* (E,π,M), *then its components satisfy the law of transformation*

$$
(4.7')\qquad (\rho,\eta)\Gamma^{a'}_{\gamma'}=M^{a'}_a\circ\pi\left[\rho^i_\gamma\circ(h\circ\pi)\frac{\partial M^a_{b'}\circ\pi}{\partial x^i}y^{b'}+(\rho,\eta)\Gamma^a_\gamma\right]\Lambda^\gamma_{\gamma'}\circ(h\circ\pi).
$$

If $\rho\Gamma$ *is a* ρ*-connection for the vector bundle* (E,π,M) *and* $h=Id_M$, *then relations* $(4.7')$ *become*

$$
(4.7'')\qquad \rho\Gamma^{a'}_{\gamma'}=M^{a'}_a\circ\pi\left[\rho^i_\gamma\circ\pi\frac{\partial M^a_{b'}\circ\pi}{\partial x^i}y^{b'}+\rho\Gamma^a_\gamma\right]\Lambda^\gamma_{\gamma'}\circ\pi.
$$

In particular, if $\rho = Id_{TM}$, *then the relations* $(4.7'')$ *become*

$$(4.7''')\qquad \Gamma_{\acute{k}}^{\acute{\imath}} = \frac{\partial x^{\acute{\imath}}}{\partial x^{i}}\circ\pi\left[\frac{\partial}{\partial x^{k}}\left(\frac{\partial x^{i}}{\partial x^{\acute{\jmath}}}\circ\pi\right)y^{\acute{\jmath}}+\Gamma_{k}^{i}\right]\frac{\partial x^{k}}{\partial x^{\acute{k}}}\circ\pi.$$

Proof. Let (Π,π) be the canonical projection **B**-morphism.

Obviously, the components of

$$\Pi\circ(\rho,\eta)\,\Gamma\left(\tilde{Z}^{\acute{\alpha}}\frac{\partial}{\partial\tilde{z}^{\acute{\alpha}}}+Y^{\acute{a}}\frac{\partial}{\partial\tilde{y}^{\acute{a}}}\right)(u_x)$$

are the real numbers

$$\left(Y^{\acute{a}}+(\rho,\eta)\,\Gamma_{\acute{\gamma}}^{\acute{a}}\tilde{Z}^{\acute{\gamma}}\right)(u_x)\,.$$

Since

$$\begin{aligned}\left(\tilde{Z}^{\acute{\alpha}}\frac{\partial}{\partial\tilde{z}^{\acute{\alpha}}}+Y^{\acute{a}}\frac{\partial}{\partial\tilde{y}^{\acute{a}}}\right)(u_x) &= \tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi\frac{\partial}{\partial\tilde{z}^{\alpha}}(u_x)\\ &+\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{\imath}}\circ h\circ\pi\frac{\partial y^{a}}{\partial x^{\acute{\imath}}}+\frac{\partial y^{a}}{\partial y^{\acute{a}}}Y^{\acute{a}}\right)\frac{\partial}{\partial\tilde{y}^{a}}(u_x)\,,\end{aligned}$$

it results that the components of

$$\Pi\circ(\rho,\eta)\,\Gamma\left(\tilde{Z}^{\acute{\alpha}}\frac{\partial}{\partial\tilde{z}^{\acute{\alpha}}}+Y^{\acute{a}}\frac{\partial}{\partial\tilde{y}^{\acute{a}}}\right)(u_x)$$

are the real numbers

$$\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{\imath}}\circ h\circ\pi\frac{\partial y^{a}}{\partial x^{\acute{\imath}}}+\frac{\partial y^{a}}{\partial y^{\acute{a}}}Y^{\acute{a}}+(\rho,\eta)\,\Gamma_{\alpha}^{a}\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi\right)(u_x)\,\frac{\partial y^{\acute{a}}}{\partial y^{a}},$$

where

$$\left\|\frac{\partial y^{a}}{\partial y^{\acute{a}}}\right\|=\left\|\frac{\partial y^{\acute{a}}}{\partial y^{a}}\right\|^{-1}.$$

Therefore, we have:

$$\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{\imath}}\circ h\circ\pi\frac{\partial y^{a}}{\partial x^{\acute{\imath}}}+\frac{\partial y^{a}}{\partial y^{\acute{a}}}Y^{\acute{a}}+(\rho,\eta)\,\Gamma_{\alpha}^{a}\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi\right)\frac{\partial y^{\acute{a}}}{\partial y^{a}}=Y^{\acute{a}}+(\rho,\eta)\,\Gamma_{\acute{\alpha}}^{\acute{a}}\tilde{Z}^{\acute{\alpha}}.$$

After some calculations we obtain:

$$(\rho,\eta)\,\Gamma_{\acute{\alpha}}^{\acute{a}}=\frac{\partial y^{\acute{a}}}{\partial y^{a}}\left(\rho_{\alpha}^{i}\circ(h\circ\pi)\,\frac{\partial y^{a}}{\partial x^{i}}+(\rho,\eta)\,\Gamma_{\alpha}^{a}\right)\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi.\qquad q.e.d.$$

Remark 4.1 If Γ is a classical connection for the vector bundle (E,π,M) on components Γ_{k}^{a}, then the differentiable real local functions $(\rho,\eta)\,\Gamma_{\gamma}^{a}=\left(\rho_{\gamma}^{k}\circ h\circ\pi\right)\Gamma_{k}^{a}$ are the components of a (ρ,η)-connection $(\rho,\eta)\,\Gamma$ for the vector bundle (E,π,M). This (ρ,η)-connection will be called the (ρ,η)-*connection associated to the connection* Γ.

Definition 4.5 If $(\rho,\eta)\,\Gamma$ is a (ρ,η)-connection for the vector bundle (E,π,M) and $z=z^{\alpha}t_{\alpha}\in\Gamma(F,\nu,N)$, then the application

$$(4.8)\qquad \begin{array}{ccc}\Gamma(E,\pi,M) & \xrightarrow{(\rho,\eta)D_z} & \Gamma(E,\pi,M)\\ u=u^{a}s_{a} & \longmapsto & (\rho,\eta)\,D_z u\end{array}$$

where

$$(\rho,\eta)\,D_z u=z^{\alpha}\circ h\left(\rho_{\alpha}^{i}\circ h\frac{\partial u^{a}}{\partial x^{i}}+(\rho,\eta)\,\Gamma_{\alpha}^{a}\circ u\right)s_{a}$$

will be called the *covariant* (ρ,η)*-derivative associated to linear* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$ *with respect to the section* z.

Definition 4.6 Let $(\rho,\eta)\,\Gamma$ be a (ρ,η)-connection for the fiber bundle (E,π,M). If for each local vector $(m+r)$-chart (U,s_U) and for each local vector $(n+p)$-chart (V,t_V) such that $U\cap h^{-1}(V)\neq\phi$, it exists the differentiable real functions $(\rho,\eta)\,\Gamma^a_{b\gamma}$ defined on $U\cap h^{-1}(V)$ such that

$$(\rho,\eta)\,\Gamma^a_{\gamma}\circ u=(\rho,\eta)\,\Gamma^a_{b\gamma}\cdot u^b,\forall u=u^bs_b\in\Gamma\left(E,\pi,M\right),\tag{4.9}$$

then we say that $(\rho,\eta)\,\Gamma$ *is linear.*

The differentiable real local functions $(\rho,\eta)\,\Gamma^a_{b\alpha}$ will be called the *Christoffel coefficients of linear* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$.

Proposition 4.1 *If* $(\rho,\eta)\,\Gamma$ *is a linear* (ρ,η)*-connection for the fiber bundle* (E,π,M), *then its components satisfy the law of transformation*

$$(\rho,\eta)\Gamma^{\acute{a}}_{\acute{b}\acute{\gamma}}=\frac{\partial y^{\acute{a}}}{\partial y^a}\left[\rho^k_{\gamma}\circ h\frac{\partial}{\partial x^k}\left(\frac{\partial y^a}{\partial y^{\acute{b}}}\right)+(\rho,\eta)\,\Gamma^a_{b\gamma}\frac{\partial y^b}{\partial y^{\acute{b}}}\right]\Lambda^{\gamma}_{\acute{\gamma}}\circ h.\tag{4.10}$$

If $(\rho,\eta)\,\Gamma$ *is a linear* (ρ,η)*-connection for the vector bundle* (E,π,M), *then its components satisfy the law of transformation*

$$(\rho,\eta)\Gamma^{\acute{a}}_{\acute{b}\acute{\gamma}}=M^{\acute{a}}_a\left[\rho^k_{\gamma}\circ h\frac{\partial M^a_{\acute{b}}}{\partial x^k}+(\rho,\eta)\Gamma^a_{b\gamma}M^b_{\acute{b}}\right]\Lambda^{\gamma}_{\acute{\gamma}}\circ h.\tag{4.10$'$}$$

If $\rho\Gamma$ *is a* ρ*-connection for the vector bundle* (E,π,M) *and* $h=Id_M$, *then the relations* (4.10$'$) *become*

$$\rho\Gamma^{\acute{a}}_{\acute{b}\acute{\gamma}}=M^{\acute{a}}_a\left[\rho^k_{\gamma}\frac{\partial M^a_{\acute{b}}}{\partial x^k}+\rho\Gamma^a_{b\gamma}M^b_{\acute{b}}\right]\Lambda^{\gamma}_{\acute{\gamma}}.\tag{4.10$''$}$$

In particular, if $\rho=Id_{TM}$, *then the relations* (4.10$''$) *become*

$$\Gamma^{\acute{i}}_{\acute{j}\acute{k}}=\frac{\partial x^{\acute{i}}}{\partial x^i}\left[\frac{\partial}{\partial x^k}\left(\frac{\partial x^i}{\partial x^{\acute{j}}}\right)+\Gamma^i_{jk}\frac{\partial x^j}{\partial x^{\acute{j}}}\right]\frac{\partial x^k}{\partial x^{\acute{k}}}.\tag{4.10$'''$}$$

Remark 4.2 If $(\rho,\eta)\,\Gamma$ is a linear (ρ,η)-connection for the vector bundle (E,π,M), then, for any

$$z=z^{\alpha}t_{\alpha}\in\Gamma\left(F,\nu,N\right),$$

we obtain the the *covariant* (ρ,η)*-derivative associated to linear* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$ *with respect to the section* z

$$\begin{array}{ccc}\Gamma\left(E,\pi,M\right)&\xrightarrow{(\rho,\eta)D_z}&\Gamma\left(E,\pi,M\right)\\ u{=}u^as_a&\longmapsto&(\rho,\eta)D_zu\end{array}\tag{4.11}$$

defined by

$$(\rho,\eta)D_zu{=}z^{\alpha}\circ h\left(\rho^i_{\alpha}\circ h\frac{\partial u^a}{\partial x^i}{+}(\rho,\eta)\Gamma^a_{b\alpha}\cdot u^b\right)s_a.$$

Remark 4.3 If $(\rho,\eta)\,\Gamma$ is a linear (ρ,η)-connection for the vector bundle (E,π,M), then the tensor fields algebra $(\mathcal{T}\left(E,\pi,M\right),+,\cdot,\otimes)$ is endowed with the (ρ,η)-derivative

$$\begin{array}{ccc}\Gamma\left(F,\nu,N\right)\times\mathcal{T}\left(E,\pi,M\right)&\xrightarrow{(\rho,\eta)D}&\mathcal{T}\left(E,\pi,M\right)\\ (z,T)&\longmapsto&(\rho,\eta)\,D_zT\end{array}\tag{4.12}$$

defined for a tensor field $T\in\mathcal{T}_q^p\left(E,\pi,M\right)$ by the relation:

$$
\begin{array}{ll}
 & (\rho,\eta)\,D_zT\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)=\Gamma\left(\rho,\eta\right)(z)\left(T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)\right)\\
(4.13) & -T\left((\rho,\eta)\,D_z\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)-...-T\left(\overset{*}{u}_1,...,(\rho,\eta)\,D_z\overset{*}{u}_p,u_1,...,u_q\right)\\
 & -T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,(\rho,\eta)\,D_zu_1,...,u_q\right)-...-T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,(\rho,\eta)\,D_zu_q\right).
\end{array}
$$

After some calculations, we obtain:

$$
\begin{array}{ll}
 & (\rho,\eta)\,D_z\left(T_{b_1,...,b_q}^{a_1,...,a_p}s_{a_1}\otimes...\otimes s_{a_p}\otimes s^{b_1}\otimes...\otimes s^{b_q}\right)\\
 & =z^\alpha\circ h\left(\rho_\alpha^i\circ h\frac{\partial T_{b_1,...,b_q}^{a_1,...,a_p}}{\partial x^i}+(\rho,\eta)\,\Gamma_{a\alpha}^{a_1}T_{b_1,...,b_q}^{a,a_2...,a_p}\right.\\
(4.14) & +(\rho,\eta)\,\Gamma_{a\alpha}^{a_2}T_{b_1,...,b_q}^{a_1,a...,a_p}+...+(\rho,\eta)\,\Gamma_{a\alpha}^{a_p}T_{b_1,...,b_q}^{a_1,a_2...,a}-...\\
 & -(\rho,\eta)\,\Gamma_{b_1\alpha}^{b}T_{b,b_2,...,b_q}^{a_1,a_2...,a_p}-(\rho,\eta)\,\Gamma_{b_2\alpha}^{b}T_{b_1,b,...,b_q}^{a_1,a_2...,a_p}-...\\
 & \left.-(\rho,\eta)\,\Gamma_{b_q\alpha}^{b}T_{b_1,b_2,...,b}^{a_1,a_2...,a_p}\right)s_{a_1}\otimes...\otimes s_{a_p}\otimes s^{b_1}\otimes...\otimes s^{b_q}\\
 & \overset{put}{=}z^\alpha\circ h\cdot T_{b_1,...,b_q|\alpha}^{a_1,...,a_p}s_{a_1}\otimes...\otimes s_{a_p}\otimes s^{b_1}\otimes...\otimes s^{b_q}.
\end{array}
$$

If $(\rho,\eta)\,\Gamma$ is the linear (ρ,η)-connection associated to linear connection Γ, namely $(\rho,\eta)\,\Gamma_{b\alpha}^a=\left(\rho_\alpha^k\circ h\right)\Gamma_{bk}^a$, then

$$
T_{b_1,...,b_q|\alpha}^{a_1,...,a_p}=\left(\rho_\alpha^k\circ h\right)T_{b_1,...,b_q|k}^{a_1,...,a_p}. \tag{4.15}
$$

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ALGEBRAIC CONSTRUCTIONS IN THE CATEGORY OF VECTOR BUNDLES

by
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Abstract

In this paper we present the category of generalized Lie algebroids. Important results (a theorem of Maurer-Cartan type, theorems of Cartan type,...) emphasize the importance and the utility of the objects of this new category.

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Contents

1 Introduction

The motivation for our researches was to extend the notion of Lie algebroid using the extension **Mod**-morphism associated to a $\mathbf{B}^{\mathbf{v}}$-morphism. Using this general framework, we get a panoramic view over classical concepts from mathematics. [1]

We introduced the notion of *interior differential system* (IDS) of a generalized Lie algebroid, in general, and of a Lie algebroid, in particular. We develop the exterior differential calculus for generalized Lie algebroids and, in this general framework, we establish the structure equations of Maurer-Cartan type and we characterize the involutivity of an IDS in a theorem of Cartan type. Finally, using the classical notion of *exterior differential system*(see: [2,4,6,7]) (EDS) of a generalized Lie algebroid, in general, and of a Lie algebroid, in particular, we characterize the involutivity of an IDS in a theorem of Cartan type. In particular, we can obtain similar results with classical results for Lie algebroids. (see: [3,8,9])

2 Preliminaries

In general, if $\mathcal{C}$ is a category, then we denoted by $|\mathcal{C}|$ the class of objects and for any $A, B \in |\mathcal{C}|$, we denote by $\mathcal{C}(A, B)$ the set of morphisms of A source and B target.

Let **Vect**, **Liealg**, **Mod, Man**, **B** and $\mathbf{B^v}$ be the category of real vector spaces, Lie algebras, modules, manifolds, fiber bundles and vector bundles respectively.

We know that if $(E, \pi, M) \in |\mathbf{B^v}|$, then $(\Gamma(E, \pi, M), +, \cdot)$ is a $\mathcal{F}(M)$-module.

In adition, if $(E, \pi, M) \in |\mathbf{B^v}|$ such that M is paracompact and if $A \subseteq M$ is closed, then for any section u over A it exists $\tilde{u} \in \Gamma(E, \pi, M)$ such that $\tilde{u}_{|A} = u$.

Note: In the following, we consider only vector bundles with paracompact base.

Proposition 2.1 If $(\varphi, \varphi_0) \in \mathbf{B^v}((E, \pi, M), (E', \pi', M'))$, then it exists a **Mod**-morphism $\Gamma(\varphi, \varphi_0)$ of $\Gamma(E, \pi, M)$ source and $\Gamma(E', \pi', M')$ target.

Proof. For every $y \in \varphi_0(M)$, we fixed $x_y \in M$ such that $\varphi_0(x_y) = y$. Then we obtain a section $\Gamma(\varphi, \varphi_0) u$ over the closed set $\varphi_0(M)$ defined by

$$\Gamma(\varphi, \varphi_0) u(y) = \varphi(u_{x_y}).$$

As M' is paracompact, then it results that the section $\Gamma(\varphi, \varphi_0) u$ can be regarded as a section of $(\Gamma(E', \pi', M') +, \cdot)$.

So, we obtain a **Mod**-morphism

$$\begin{array}{ccc} \Gamma(E, \pi, M) & \xrightarrow{\Gamma(\varphi, \varphi_0)} & \Gamma(E', \pi', M') \\ u & \longmapsto & \Gamma(\varphi, \varphi_0) u \end{array}$$

defined by

$$\Gamma(\varphi, \varphi_0) u(y) = \varphi(u_{x_y}),$$

for any $y \in \varphi_0(M)$. *q.e.d.*

Definition 2.1 A **Mod**-morphism given by the previous proposition is called *the extension* **Mod**-*morphism associated to the* $\mathbf{B^v}$-*morphism* (φ, φ_0).

Remark 2.1 The construction of the extension **Mod**-morphism associated to a $\mathbf{B^v}$-morphism (φ, φ_0) is not unique, but any two extension **Mod**-morphisms associated to a $\mathbf{B^v}$-morphism (φ, φ_0) has the same properties.

Example 2.1 If $(\varphi, \varphi_0) \in \mathbf{B^v}((E, \pi, M), (E', \pi', M'))$ such that $\varphi_0 \in Diff(M, M')$, then we obtain the unique **Mod**-morphism

$$\begin{array}{ccc} \Gamma(E, \pi, M) & \xrightarrow{\Gamma(\varphi, \varphi_0)} & \Gamma(E', \pi', M') \\ u & \longmapsto & \Gamma(\varphi, \varphi_0) u \end{array}$$

defined by

$$(\Gamma(\varphi, \varphi_0) u)(x') = \varphi\left(u_{\varphi_0^{-1}(x')}\right).$$

$\Gamma(\varphi, \varphi_0)$ is called *the* **Mod**-*morphism associated to the* $\mathbf{B^v}$-*morphism* (φ, φ_0).

3 The category of generalized Lie algebroids

We know that a Lie algebroid is a vector bundle $(F, \nu, N) \in |\mathbf{B^v}|$ such that there exists

$$(\rho, Id_N) \in \mathbf{B^v}((F, \nu, N), (TN, \tau_N, N))$$

and an operation

$$\begin{array}{ccc}\Gamma(F,\nu,N)\times\Gamma(F,\nu,N) & \xrightarrow{[,]_F} & \Gamma(F,\nu,N)\\ (u,v) & \longmapsto & [u,v]_F\end{array}$$

with the following properties:

LA_1. the equality holds good

$$[u,f\cdot v]_F=f\,[u,v]_F+\Gamma(\rho,Id_N)(u)\,f\cdot v,$$

for all $u,v\in\Gamma(F,\nu,N)$ and $f\in\mathcal{F}(N)$,

LA_2. the 4-tuple $(\Gamma(F,\nu,N),+,\cdot,[,]_F)$ is a Lie $\mathcal{F}(N)$-algebra,

LA_3. the **Mod**-morphism $\Gamma(\rho,Id_N)$ is a **LieAlg**-morphism of $(\Gamma(F,\nu,N),+,\cdot,[,]_F)$ source and $(\Gamma(TN,\tau_N,N),+,\cdot,[,]_{TN})$ target.

Obviously, in the definition of the Lie algebroid we use essential the **Mod**-morphism $\Gamma(\rho,Id_N)$ associated to the $\mathbf{B^v}$-morphism (ρ,Id_N).

So, we are interested to finding the answer to the following question:

- *Could we to extend the notion of Lie algebroid using the extension* **Mod**-*morphism associated to a* $\mathbf{B^v}$-*morphism?*

Definition 3.1 Let $M,N\in|\mathbf{Man}|$ and $h\in\mathbf{Man}(M,N)$ be surjective.
If $(F,\nu,N)\in|\mathbf{B^v}|$ such that there exists

$$(\rho,\eta)\in\mathbf{B^v}((F,\nu,N),(TM,\tau_M,M))$$

and an operation

$$\begin{array}{ccc}\Gamma(F,\nu,N)\times\Gamma(F,\nu,N) & \xrightarrow{[,]_{F,h}} & \Gamma(F,\nu,N)\\ (u,v) & \longmapsto & [u,v]_{F,h}\end{array}$$

with the following properties:

GLA_1. the equality holds good

$$[u,f\cdot v]_{F,h}=f\,[u,v]_{F,h}+\Gamma(Th\circ\rho,h\circ\eta)(u)\,f\cdot v,$$

for all $u,v\in\Gamma(F,\nu,N)$ and $f\in\mathcal{F}(N)$.

GLA_2. the 4-tuple $\left(\Gamma(F,\nu,N),+,\cdot,[,]_{F,h}\right)$ is a Lie $\mathcal{F}(N)$-algebra,

GLA_3. the **Mod**-morphism $\Gamma(Th\circ\rho,h\circ\eta)$ is a **LieAlg**-morphism of $\left(\Gamma(F,\nu,N),+,\cdot,[,]_{F,h}\right)$ source and $(\Gamma(TN,\tau_N,N),+,\cdot,[,]_{TN})$ target, then we will say that *the triple*

$$\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)\tag{3.1}$$

is a generalized Lie algebroid. The couple $\left([,]_{F,h},(\rho,\eta)\right)$ will be called *generalized Lie algebroid structure.*

Definition 3.2 We define the set of morphisms of

$$\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$$

source and

$$\left(\left(F^{\prime},\nu^{\prime},N^{\prime}\right),\left[,\right]_{F^{\prime},h^{\prime}},\left(\rho^{\prime},\eta^{\prime}\right)\right)$$

target as being the set

$$\left\{\left(\varphi,\varphi_{0}\right)\in\mathbf{B^{v}}\left(\left(F,\nu,N\right),\left(F^{\prime},\nu^{\prime},N^{\prime}\right)\right)\right\}$$

such that the **Mod**-morphism $\Gamma\left(\varphi,\varphi_{0}\right)$ is a **LieAlg**-morphism of

$$\left(\Gamma\left(F,\nu,N\right),+,\cdot,\left[,\right]_{F,h}\right)$$

source and

$$\left(\Gamma\left(F^{\prime},\nu^{\prime},N^{\prime}\right),+,\cdot,\left[,\right]_{F^{\prime},h^{\prime}}\right)$$

target.

We remark that we can discuss about *the category* **GLA** *of generalized Lie algebroids.*

Let $\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$ be a generalized Lie algebroid.

- Locally, for any $\alpha,\beta\in\overline{1,p}$, we set $\left[t_{\alpha},t_{\beta}\right]_{F}\overset{put}{=}L_{\alpha\beta}^{\gamma}t_{\gamma}$. We easily obtain that $L_{\alpha\beta}^{\gamma}=-L_{\beta\alpha}^{\gamma}$, for any $\alpha,\beta,\gamma\in\overline{1,p}$.

The real local functions $\left\{L_{\alpha\beta}^{\gamma},\ \alpha,\beta,\gamma\in\overline{1,p}\right\}$ will be called the *structure functions of the generalized Lie algebroid* $\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$.

- We assume that $\left(F,\nu,N\right)$ is a vector bundle with type fibre the real vector space $\left(\mathbb{R}^{p},+,\cdot\right)$ and structure group a Lie subgroup of $\left(\mathbf{GL}\left(p,\mathbb{R}\right),\cdot\right)$.

We take $\left(x^{i},y^{i}\right)$ as canonical local coordinates on $\left(TM,\tau_{M},M\right)$, where $i\in\overline{1,m}$.

Consider $\left(x^{i},y^{i}\right)\longrightarrow\left(x^{i\acute{}}\left(x^{i}\right),y^{i\acute{}}\left(x^{i},y^{i}\right)\right)$ a change of coordinates on $\left(TM,\tau_{M},M\right)$. Then the coordinates y^{i} change to $y^{i\acute{}}$ by the rule:

$$y^{i\acute{}}=\tfrac{\partial x^{i\acute{}}}{\partial x^{i}}y^{i}. \tag{3.2}$$

We take $\left(\varkappa^{\tilde{\imath}},z^{\alpha}\right)$ as canonical local coordinates on $\left(F,\nu,N\right)$, where $\tilde{\imath}\in\overline{1,n}$, $\alpha\in\overline{1,p}$.

Consider $\left(\varkappa^{\tilde{\imath}},z^{\alpha}\right)\longrightarrow\left(\varkappa^{\tilde{\imath}\acute{}},z^{\alpha\acute{}}\right)$ a change of coordinates on $\left(F,\nu,N\right)$. Then the coordinates z^{α} change to $z^{\alpha\acute{}}$ by the rule:

$$z^{\alpha\acute{}}=\Lambda_{\alpha}^{\alpha\acute{}}z^{\alpha}. \tag{3.3}$$

- We assume that $\left(\theta,\mu\right)\overset{put}{=}\left(Th\circ\rho,h\circ\eta\right)$. If $z^{\alpha}t_{\alpha}\in\Gamma\left(F,\nu,N\right)$ is arbitrary, then

$$\begin{array}{c}\Gamma\left(Th\circ\rho,h\circ\eta\right)\left(z^{\alpha}t_{\alpha}\right)f\left(h\circ\eta\left(\varkappa\right)\right)=\\ =\left(\theta_{\alpha}^{\tilde{\imath}}z^{\alpha}\dfrac{\partial f}{\partial\varkappa^{\tilde{\imath}}}\right)\left(h\circ\eta\left(\varkappa\right)\right)=\left(\left(\rho_{\alpha}^{i}\circ h\right)\left(z^{\alpha}\circ h\right)\dfrac{\partial f\circ h}{\partial x^{i}}\right)\left(\eta\left(\varkappa\right)\right),\end{array} \tag{3.4}$$

for any $f\in\mathcal{F}\left(N\right)$ and $\varkappa\in N$.

The coefficients ρ_{α}^{i} respectively $\theta_{\alpha}^{\tilde{i}}$ change to $\rho_{\alpha'}^{i'}$ respectively $\theta_{\alpha'}^{\tilde{i}'}$ by the rule:

$$\rho_{\alpha'}^{i'} = \Lambda_{\alpha'}^{\alpha}\rho_{\alpha}^{i}\frac{\partial x^{i'}}{\partial x^{i}}, \tag{3.5}$$

respectively

$$\theta_{\alpha'}^{\tilde{i}'} = \Lambda_{\alpha'}^{\alpha}\theta_{\alpha}^{\tilde{i}}\frac{\partial \varkappa^{\tilde{i}'}}{\partial \varkappa^{\tilde{i}}}, \tag{3.6}$$

where $\|\Lambda_{\alpha'}^{\alpha}\| = \left\|\Lambda_{\alpha}^{\alpha'}\right\|^{-1}$.

Remark 3.1 The following equalities hold good:

$$\rho_{\alpha}^{i}\circ h\frac{\partial f\circ h}{\partial x^{i}} = \left(\theta_{\alpha}^{\tilde{i}}\frac{\partial f}{\partial \varkappa^{\tilde{i}}}\right)\circ h, \forall f\in \mathcal{F}(N). \tag{3.7}$$

and

$$\left(L_{\alpha\beta}^{\gamma}\circ h\right)\left(\rho_{\gamma}^{k}\circ h\right) = \left(\rho_{\alpha}^{i}\circ h\right)\frac{\partial\left(\rho_{\beta}^{k}\circ h\right)}{\partial x^{i}} - \left(\rho_{\beta}^{j}\circ h\right)\frac{\partial\left(\rho_{\alpha}^{k}\circ h\right)}{\partial x^{j}}. \tag{3.8}$$

In the next we build some examples of objects of the category **GLA**.

Theorem 3.1 *Let $M\in|\mathbf{Man}_m|$ and $g,h\in Iso_{\mathbf{Man}}(M)$ be. Using the tangent $\mathbf{B^v}$-morphism (Tg,g) and the operation*

$$\begin{array}{ccc} \Gamma(TM,\tau_M,M)\times\Gamma(TM,\tau_M,M) & \xrightarrow{[,]_{TM,h}} & \Gamma(TM,\tau_M,M)\\ (u,v) & \longmapsto & [u,v]_{TM,h}\end{array}$$

where

$$[u,v]_{TM,h} = \Gamma\left(T(h\circ g)^{-1},(h\circ g)^{-1}\right)\left([\Gamma(T(h\circ g),h\circ g)u,\Gamma(T(h\circ g),h\circ g)v]_{TM}\right),$$

for any $u,v\in\Gamma(TM,\tau_M,M)$, we obtain that

$$\left((TM,\tau_M,M),(Tg,g),[,]_{TM,h}\right)\in|\mathbf{GLA}|.$$

For any **Man**-isomorphisms g and h we obtain new and interesting generalized Lie algebroid structures for the tangent vector bundle (TM,τ_M,M). For any base $\left\{t_{\alpha},\ \alpha\in\overline{1,m}\right\}$ of the module of sections $(\Gamma(TM,\tau_M,M),+,\cdot)$ we obtain the structure functions

$$L_{\alpha\beta}^{\gamma} = \left(\theta_{\alpha}^{i}\frac{\partial\theta_{\beta}^{j}}{\partial x^{i}} - \theta_{\beta}^{i}\frac{\partial\theta_{\alpha}^{j}}{\partial x^{i}}\right)\tilde{\theta}_{j}^{\gamma},\ \alpha,\beta,\gamma\in\overline{1,m}$$

where $\theta_{\alpha}^{i},\ i,\alpha\in\overline{1,m}$ are real local functions such that

$$\Gamma(T(h\circ g),h\circ g)(t_{\alpha}) = \theta_{\alpha}^{i}\frac{\partial}{\partial x^{i}}$$

and $\tilde{\theta}_{j}^{\gamma},\ i,\gamma\in\overline{1,m}$ are real local functions such that

$$\Gamma\left(T(h\circ g)^{-1},(h\circ g)^{-1}\right)\left(\frac{\partial}{\partial x^{j}}\right) = \tilde{\theta}_{j}^{\gamma}t_{\gamma}.$$

In particular, using arbitrary basis for the module of sections and arbitrary isometries (symmetries, translations, rotations,...) for the Euclidean 3-dimensional space Σ,

we obtain a lot of generalized Lie algebroid structures for the tangent vector bundle $(T\Sigma, \tau_{\Sigma}, \Sigma)$.

We assume that $((F,\nu,N),[,]_F,(\rho,Id_N))$ is a Lie algebroid and let $h\in\mathbf{Man}(N,N)$ be a surjective application. Let $\mathcal{AF}_F$ be a vector fibred $(n+p)$-atlas for the vector bundle (F,ν,N) and let $\mathcal{AF}_{TN}$ be a vector fibred $(n+n)$-atlas for the vector bundle (TN,τ_N,N).

If $(U,\xi_U)\in\mathcal{AF}_{TN}$ and $(V,s_V)\in\mathcal{AF}_F$ such that $U\cap h^{-1}(V)\neq\phi$, then we define the application

$$\begin{array}{ccc}\tau_N^{-1}(U\cap h^{-1}(V))) & \xrightarrow{\bar{\xi}_{U\cap h^{-1}(V)}} & \left(U\cap h^{-1}(V)\right)\times\mathbb{R}^n\\ (\varkappa,u(\varkappa)) & \longmapsto & \left(\varkappa,\xi_{U,\varkappa}^{-1}u(\varkappa)\right).\end{array}$$

Proposition 3.1 *The set*

$$\overline{\mathcal{AF}}_{TN}\overset{put}{=}\bigcup_{\substack{(U,\xi_U)\in\mathcal{AF}_{TN},\ (V,s_V)\in\mathcal{AF}_F\\ U\cap h^{-1}(V)\neq\phi}}\left\{\left(U\cap h^{-1}(V),\bar{\xi}_{U\cap h^{-1}(V)}\right)\right\}$$

is a vector fibred $n+n$-atlas for the vector bundle (TN,τ_N,N).

If $X=X^{\tilde{\imath}}\frac{\partial}{\partial\varkappa^{\tilde{\imath}}}\in\Gamma(TN,\tau_N,N)$, then we obtain the section

$$\bar{X}=\bar{X}^{\tilde{\imath}}\circ h\frac{\partial}{\partial\bar{\varkappa}^{\tilde{\imath}}}\in\Gamma(TN,\tau_N,N),$$

such that $\bar{X}(\bar{\varkappa})=X(h(\bar{\varkappa}))$, for any $\bar{\varkappa}\in U\cap h^{-1}(V)$.

The set $\left\{\frac{\partial}{\partial\bar{\varkappa}^{\tilde{\imath}}},\ \tilde{\imath}\in\overline{1,n}\right\}$ is a base for the $\mathcal{F}(N)$-module $(\Gamma(TN,\tau_N,N),+,\cdot)$.

Theorem 3.2 *If we consider the operation*

$$\Gamma(F,\nu,N)\times\Gamma(F,\nu,N)\xrightarrow{[,]_{F,h}}\Gamma(F,\nu,N)$$

defined by

$$\begin{aligned}[][t_\alpha,t_\beta]_{F,h} &= \left(L_{\alpha\beta}^{\gamma}\circ h\right)t_\gamma,\\ [t_\alpha,ft_\beta]_{F,h} &= f\left(L_{\alpha\beta}^{\gamma}\circ h\right)t_\gamma+\rho_\alpha^{\tilde{\imath}}\circ h\frac{\partial f}{\partial\bar{\varkappa}^{\tilde{\imath}}}t_\beta,\\ [ft_\alpha,t_\beta]_{F,h} &= -[t_\beta,ft_\alpha]_{F,h},\end{aligned}$$

for any $f\in\mathcal{F}(N)$, then $\left((F,\nu,N),[,]_{F,h},(\rho,Id_N)\right)\in|\mathbf{GLA}|$.

The generalized Lie algebroid

$$\left((F,\nu,N),[,]_{F,h},(\rho,Id_N)\right)$$

given by the previous theorem, will be called *the generalized Lie algebroid associated to the Lie algebroid $((F,\nu,N),[,]_F,(\rho,Id_N))$ and to the surjective application $h\in$* $\mathbf{Man}(N,N)$.

In particular, if $h=Id_N$, then the generalized Lie algebroid

$$\left((F,\nu,N),[,]_{F,Id_N},(\rho,Id_N)\right)$$

will be called *the generalized Lie algebroid associated to the Lie algebroid*

$$((F,\nu,N),[,]_F,(\rho,Id_N)).$$

3.1 The pull-back Lie algebroid of a generalized Lie algebroid

Let $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ be a generalized Lie algebroid.

Let $\mathcal{AF}_F$ be a vector fibred $(n+p)$-atlas for the vector bundle (F,ν,N) and let $\mathcal{AF}_{TM}$ be a vector fibred $(m+m)$-atlas for the vector bundle (TM,τ_M,M).

Let $(h^*F,h^*\nu,M)$ be the pull-back vector bundle through h.

If $(U,\xi_U)\in\mathcal{AF}_{TM}$ and $(V,s_V)\in\mathcal{AF}_F$ such that $U\cap h^{-1}(V)\neq\phi$, then we define the application

$$\begin{array}{ccc} h^*\nu^{-1}(U\cap h^{-1}(V))) & \xrightarrow{\bar{s}_{U\cap h^{-1}(V)}} & \left(U\cap h^{-1}(V)\right)\times\mathbb{R}^p \\ (\varkappa,z(h(\varkappa))) & \longmapsto & \left(\varkappa,t_{V,h(\varkappa)}^{-1}z(h(\varkappa))\right). \end{array}$$

Proposition 3.1.1 *The set*

$$\overline{\mathcal{AF}}_F\overset{put}{=}\bigcup_{\substack{(U,\xi_U)\in\mathcal{AF}_{TM},\ (V,s_V)\in\mathcal{AF}_F\\ U\cap h^{-1}(V)\neq\phi}}\left\{\left(U\cap h^{-1}(V),\bar{s}_{U\cap h^{-1}(V)}\right)\right\}$$

is a vector fibred $m+p$-atlas for the vector bundle $(h^*F,h^*\nu,M)$.

If $z=z^\alpha t_\alpha\in\Gamma(F,\nu,N)$, then we obtain the section

$$Z=(z^\alpha\circ h)T_\alpha\in\Gamma(h^*F,h^*\nu,M)$$

such that $Z(x)=z(h(x))$, for any $x\in U\cap h^{-1}(V)$.

Theorem 3.1.1 *Let $\left(\overset{h^*F}{\rho},Id_M\right)$ be the $\mathbf{B^v}$-morphism of $(h^*F,h^*\nu,M)$ source and (TM,τ_M,M) target, where*

$$\begin{array}{ccc} h^*F & \xrightarrow{\overset{h^*F}{\rho}} & TM \\ Z^\alpha T_\alpha(x) & \longmapsto & \left(Z^\alpha\cdot\rho_\alpha^i\circ h\right)\dfrac{\partial}{\partial x^i}(x) \end{array}\tag{3.1.1}$$

Using the operation

$$\Gamma(h^*F,h^*\nu,M)\times\Gamma(h^*F,h^*\nu,M)\xrightarrow{[,]_{h^*F}}\Gamma(h^*F,h^*\nu,M)$$

defined by

$$\begin{array}{ll} [T_\alpha,T_\beta]_{h^*F} & =\left(L_{\alpha\beta}^\gamma\circ h\right)T_\gamma, \\ [T_\alpha,fT_\beta]_{h^*F} & =f\left(L_{\alpha\beta}^\gamma\circ h\right)T_\gamma+\left(\rho_\alpha^i\circ h\right)\dfrac{\partial f}{\partial x^i}T_\beta, \\ [fT_\alpha,T_\beta]_{h^*F} & =-[T_\beta,fT_\alpha]_{h^*F}, \end{array}\tag{3.1.2}$$

for any $f\in\mathcal{F}(M)$, it results that

$$\left((h^*F,h^*\nu,M),[,]_{h^*F},\left(\overset{h^*F}{\rho},Id_M\right)\right)$$

ia a Lie algebroid which is called *the pull-back Lie algebroid of the generalized Lie algebroid* $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$.

3.2 Interior Differential Systems

Let $\left((h^*F, h^*\nu, M), [,]_{h^*F}, \left(\overset{h^*F}{\rho}, Id_M\right)\right)$ be the pull-back Lie algebroid of the generalized Lie algebroid $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$.

Definition 3.2.1 Any vector subbundle (E, π, M) of the vector bundle $(h^*F, h^*\nu, M)$ will be called *interior differential system (IDS) of the generalized Lie algebroid*

$$\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right).$$

In particular, if $h = Id_N = \eta$, then any vector subbundle (E, π, N) of the vector bundle (F, ν, N) will be called *interior differential system of the Lie algebroid*

$$((F, \nu, N), [,]_F, (\rho, Id_N)).$$

Remark 3.2.1 If (E, π, M) is an IDS of the generalized Lie algebroid

$$\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right),$$

then we obtain a vector subbundle $\left(E^0, \pi^0, M\right)$ of the vector bundle $\left(\overset{*}{h}F, \overset{*}{h}\nu, M\right)$ such that

$$\Gamma\left(E^0, \pi^0, M\right) \overset{put}{=} \left\{\Omega \in \Gamma\left(\overset{*}{h}F, \overset{*}{h}\nu, M\right) : \Omega(S) = 0,\ \forall S \in \Gamma(E, \pi, M)\right\}.$$

The vector subbundle $\left(E^0, \pi^0, M\right)$ will be called *the annihilator vector subbundle of the IDS* (E, π, M).

Proposition 3.2.1 *If* (E, π, M) *is an IDS of the generalized Lie algebroid*

$$\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$$

such that $\Gamma(E, \pi, M) = \langle S_1, ..., S_r\rangle$, *then it exists* $\Theta^{r+1}, ..., \Theta^p \in \Gamma\left(\overset{*}{h}F, \overset{*}{h}\nu, M\right)$ *linearly independent such that* $\Gamma\left(E^0, \pi^0, M\right) = \left\langle \Theta^{r+1}, ..., \Theta^p\right\rangle$.

Definition 3.2.2 The IDS (E, π, M) of the generalized Lie algebroid

$$\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$$

will be called *involutive* if $[S, T]_{h^*F} \in \Gamma(E, \pi, M)$, for any $S, T \in \Gamma(E, \pi, M)$.

Proposition 3.2.2 *If* (E, π, M) *is an IDS of the generalized Lie algebroid*

$$\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$$

and $\{S_1, ..., S_r\}$ *is a base for the* $\mathcal{F}(M)$*-submodule* $(\Gamma(E, \pi, M), +, \cdot)$ *then* (E, π, M) *is involutive if and only if* $[S_a, S_b]_{h^*F} \in \Gamma(E, \pi, M)$, for any $a, b \in \overline{1, r}$.

4 Exterior differential calculus

Let $\left((F,\nu,N), [,]_{F,h}, (\rho,\eta) \right)$ be a generalized Lie algebroid. We denoted by $\Lambda^q(F,\nu,N)$ the set of *differential forms of degree* q. We remark that if

$$\Lambda(F,\nu,N) = \underset{q\geq 0}{\oplus} \Lambda^q(F,\nu,N),$$

then we obtain *the exterior differential algebra* $(\Lambda(F,\nu,N), +, \cdot, \wedge)$.

Definition 4.1 For any $z \in \Gamma(F,\nu,N)$, the application

$$\Lambda(F,\nu,N) \xrightarrow{L_z} \Lambda(F,\nu,N),$$

defined by

$$L_z(f) = \Gamma(Th\circ\rho, h\circ\eta)\, z(f),$$

for any $f \in \mathcal{F}(N)$ and

$$\begin{array}{ll} L_z\omega(z_1,...,z_q) & = \Gamma(Th\circ\rho, h\circ\eta)\, z(\omega((z_1,...,z_q))) \\ & -\sum\limits_{i=1}^{q}\omega\left(\left(z_1,...,[z,z_i]_{F,h},...,z_q\right)\right), \end{array}$$

for any $\omega \in \Lambda^q(F,\nu,N)$ and $z_1,...,z_q \in \Gamma(F,\nu,N)$, is called *the covariant Lie derivative with respect to the section* z.

Theorem 4.1 *If* $z \in \Gamma(F,\nu,N)$, $\omega \in \Lambda^q(F,\nu,N)$ *and* $\theta \in \Lambda^r(F,\nu,N)$, *then*

$$L_z(\omega\wedge\theta) = L_z\omega\wedge\theta + \omega\wedge L_z\theta. \tag{4.1}$$

Definition 4.2 For any $z \in \Gamma(F,\nu,N)$, the application

$$\begin{array}{rcl} \Lambda(F,\nu,N) & \xrightarrow{i_z} & \Lambda(F,\nu,N) \\ \Lambda^q(F,\nu,N) \ni \omega & \longmapsto & i_z\omega \in \Lambda^{q-1}(F,\nu,N), \end{array}$$

defined by $i_z f = 0$, for any $f \in \mathcal{F}(N)$ and

$$i_z\omega(z_2,...,z_q) = \omega(z,z_2,...,z_q),$$

for any $z_2,...,z_q \in \Gamma(F,\nu,N)$, is called the *interior product associated to the section* z.

Theorem 4.2 *If* $z \in \Gamma(F,\nu,N)$, *then for any* $\omega \in \Lambda^q(F,\nu,N)$ *and* $\theta \in \Lambda^r(F,\nu,N)$ *we obtain*

$$i_z(\omega\wedge\theta) = i_z\omega\wedge\theta + (-1)^q\omega\wedge i_z\theta. \tag{4.2}$$

Theorem 4.3 *For any* $z,v \in \Gamma(F,\nu,N)$ *we obtain*

$$L_v\circ i_z - i_z\circ L_v = i_{[z,v]_{F,h}}. \tag{4.3}$$

Theorem 4.4 *The application*

$$\begin{array}{ccc} \Lambda^q(F,\nu,N) & \xrightarrow{d^F} & \Lambda^{q+1}(F,\nu,N) \\ \omega & \longmapsto & d\omega \end{array}$$

defined by $d^F f(z) = \Gamma(Th \circ \rho, h \circ \eta)(z) f$, *for any* $z \in \Gamma(F, \nu, N)$, *and*

$$d^F \omega(z_0, z_1, ..., z_q) = \sum_{i=0}^{q} (-1)^i \Gamma(Th \circ \rho, h \circ \eta) z_i (\omega((z_0, z_1, ..., \hat{z}_i, ..., z_q))) + \sum_{i<j} (-1)^{i+j} \omega\left(\left([z_i, z_j]_{F,h}, z_0, z_1, ..., \hat{z}_i, ..., \hat{z}_j, ..., z_q\right)\right),$$

for any $z_0, z_1, ..., z_q \in \Gamma(F, \nu, N)$, *is unique with the following property:*

$$L_z = d^F \circ i_z + i_z \circ d^F, \ \forall z \in \Gamma(F, \nu, N). \tag{4.4}$$

This application will be called *the exterior differentiation operator for the exterior differential algebra of the generalized Lie algebroid* $((F, \nu, N), [,]_{F,h}, (\rho, \eta))$.

Theorem 4.5 *The exterior differentiation operator* d^F *given by the previous theorem has the following properties:*

1. *For any* $\omega \in \Lambda^q(F, \nu, N)$ *and* $\theta \in \Lambda^r(F, \nu, N)$ *we obtain*

$$d^F(\omega \wedge \theta) = d^F \omega \wedge \theta + (-1)^q \omega \wedge d^F \theta. \tag{4.5}$$

2. *For any* $z \in \Gamma(F, \nu, N)$ *we obtain* $L_z \circ d^F = d^F \circ L_z$.
3. $d^F \circ d^F = 0$.

Theorem 4.6 (of Maurer-Cartan type)

If $((F, \nu, N), [,]_{F,h}, (\rho, \eta))$ *is a generalized Lie algebroid and* d^F *is the exterior differentiation operator for the exterior differential* $\mathcal{F}(N)$*-algebra* $(\Lambda(F, \nu, N), +, \cdot, \wedge)$, *then we obtain the structure equations of Maurer-Cartan type*

$$d^F t^\alpha = -\frac{1}{2} L^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma, \ \alpha \in \overline{1, p} \tag{$\mathcal{C}_1$}$$

and

$$d^F \varkappa^{\tilde{\imath}} = \theta^{\tilde{\imath}}_\alpha t^\alpha, \ \tilde{\imath} \in \overline{1, n}, \tag{$\mathcal{C}_2$}$$

where $\left\{t^\alpha, \alpha \in \overline{1, p}\right\}$ *is the coframe of the vector bundle* (F, ν, N).

This equations will be called *the structure equations of Maurer-Cartan type associated to the generalized Lie algebroid* $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$.

Proof. Let $\alpha \in \overline{1, p}$ be arbitrary. Since

$$d^F t^\alpha(t_\beta, t_\gamma) = -L^\alpha_{\beta\gamma}, \ \forall \beta, \gamma \in \overline{1, p}$$

it results that

$$d^F t^\alpha = -\sum_{\beta<\gamma} L^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma. \tag{1}$$

Since $L^\alpha_{\beta\gamma} = -L^\alpha_{\gamma\beta}$ and $t^\beta \wedge t^\gamma = -t^\gamma \wedge t^\beta$, for nay $\beta, \gamma \in \overline{1, p}$, it results that

$$\sum_{\beta<\gamma} L^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma = \frac{1}{2} L^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma \tag{2}$$

Using the equalities (1) and (2) it results the structure equation $(\mathcal{C}_1)$.

Let $\tilde{\imath} \in \overline{1,n}$ be arbitrarily. Since $d^F \varkappa^{\tilde{\imath}}(t_\alpha) = \theta^{\tilde{\imath}}_\alpha$, for any $\alpha \in \overline{1,p}$, it results the structure equation $(\mathcal{C}_2)$. *q.e.d.*

Corollary 4.1 *If* $\left((h^*F, h^*\nu, M), [,]_{h^*F}, \left(\overset{h^*F}{\rho}, Id_M\right)\right)$ *is the pull-back Lie algebroid associated to the generalized Lie algebroid* $\left((F,\nu,N), [,]_{F,h}, (\rho,\eta)\right)$ *and* d^{h^*F} *is the exterior differentiation operator for the exterior differential* $\mathcal{F}(M)$*-algebra* $(\Lambda(h^*F, h^*\nu, M), +, \cdot, \wedge)$*, then we obtain the following structure equations of Maurer-Cartan type*

$$(\mathcal{C}_1') \qquad d^{h^*F}T^\alpha = -\frac{1}{2}\left(L^\alpha_{\beta\gamma} \circ h\right) T^\beta \wedge T^\gamma,\ \alpha \in \overline{1,p}$$

and

$$(\mathcal{C}_2') \qquad d^{h^*F}x^i = \left(\rho^i_\alpha \circ h\right) T^\alpha,\ i \in \overline{1,m}.$$

This equations will be called *the structure equations of Maurer-Cartan type associated to the pull-back Lie algebroid*

$$\left((h^*F, h^*\nu, M), [,]_{h^*F}, \left(\overset{h^*F}{\rho}, Id_M\right)\right).$$

Theorem 4.7 (of Cartan type) *Let* (E,π,M) *be an IDS of the generalized Lie algebroid* $\left((F,\nu,N), [,]_{F,h}, (\rho,\eta)\right)$. *If* $\left\{\Theta^{r+1}, ..., \Theta^p\right\}$ *is a base for the* $\mathcal{F}(M)$*-submodule* $\left(\Gamma\left(E^0, \pi^0, M\right), +, \cdot\right)$*, then the IDS* (E,π,M) *is involutive if and only if it exists*

$$\Omega^\alpha_\beta \in \Lambda^1(h^*F, h^*\nu, M),\ \alpha,\beta \in \overline{r+1,p}$$

such that

$$d^{h^*F}\Theta^\alpha = \Sigma_{\beta\in\overline{r+1,p}}\Omega^\alpha_\beta \wedge \Theta^\beta \in \mathcal{I}\left(\Gamma\left(E^0,\pi^0,M\right)\right).$$

Proof. Let $\{S_1, ..., S_r\}$ be a base for the $\mathcal{F}(M)$-submodule $(\Gamma(E,\pi,M), +, \cdot)$

Let $\{S_{r+1}, ..., S_p\} \in \Gamma(h^*F, h^*\nu, M)$ such that $\{S_1, ..., S_r, S_{r+1}, ..., S_p\}$ is a base for the $\mathcal{F}(M)$-module

$$(\Gamma(h^*F, h^*\nu, M), +, \cdot).$$

Let $\Theta^1, ..., \Theta^r \in \Gamma\left(\overset{*}{h^*}F, \overset{*}{h^*}\nu, M\right)$ such that $\left\{\Theta^1, ..., \Theta^r, \Theta^{r+1}, ..., \Theta^p\right\}$ is a base for the $\mathcal{F}(M)$-module

$$\left(\Gamma\left(\overset{*}{h^*}F, \overset{*}{h^*}\nu, M\right), +, \cdot\right).$$

For any $a,b \in \overline{1,r}$ and $\alpha,\beta \in \overline{r+1,p}$, we have the equalities:

$$\begin{array}{rcl} \Theta^a(S_b) & = & \delta^a_b \\ \Theta^a(S_\beta) & = & 0 \\ \Theta^\alpha(S_b) & = & 0 \\ \Theta^\alpha(S_\beta) & = & \delta^\alpha_\beta \end{array}$$

We remark that the set of the 2-forms

$$\left\{\Theta^a \wedge \Theta^b, \Theta^a \wedge \Theta^\beta, \Theta^\alpha \wedge \Theta^\beta,\ a,b \in \overline{1,r} \wedge \alpha,\beta \in \overline{r+1,p}\right\}$$

is a base for the $\mathcal{F}(M)$-module $\left(\Lambda^2(h^*F, h^*\nu, M), +, \cdot\right)$.

Therefore, we have

$$d^{h^*F}\Theta^\alpha = \Sigma_{b<c}A_{bc}^\alpha\Theta^b\wedge\Theta^c + \Sigma_{b,\gamma}B_{b\gamma}^\alpha\Theta^b\wedge\Theta^\gamma + \Sigma_{\beta<\gamma}C_{\beta\gamma}^\alpha\Theta^\beta\wedge\Theta^\gamma, \tag{1}$$

where, $A_{bc}^\alpha, B_{b\gamma}^\alpha$ and $C_{\beta\gamma}^\alpha$, $a,b,c\in\overline{1,r}$, $\alpha,\beta,\gamma\in\overline{r+1,p}$ are real local functions such that $A_{bc}^\alpha = -A_{cb}^\alpha$ and $C_{\beta\gamma}^\alpha = -C_{\gamma\beta}^\alpha$.

Using the formula

$$\begin{array}{ll} d^{h^*F}\Theta^\alpha(S_b, S_c) & = \Gamma\left(\overset{h^*F}{\rho}, Id_M\right)S_b\left(\Theta^\alpha(S_c)\right) - \Gamma\left(\overset{h^*F}{\rho}, Id_M\right)S_c\left(\Theta^\alpha(S_b)\right) \\ & -\Theta^\alpha\left([S_b, S_c]_{h^*F}\right), \end{array} \tag{2}$$

we obtain that

$$A_{bc}^\alpha = -\Theta^\alpha\left([S_b, S_c]_{h^*F}\right), \tag{3}$$

for any $b,c\in\overline{1,r}$ and $\alpha\in\overline{r+1,p}$.

We admit that (E,π,M) is an involutive IDS of the generalized Lie algebroid $\left((F,\nu,N), [,]_{F,h}, (\rho,\eta)\right)$.

As $[S_b, S_c]_{h^*F}\in\Gamma(E,\pi,M)$, for any $b,c\in\overline{1,r}$, it results that $\Theta^\alpha\left([S_b, S_c]_{h^*F}\right) = 0$, for any $b,c\in\overline{1,r}$ and $\alpha\in\overline{r+1,p}$. Therefore, for any $b,c\in\overline{1,r}$ and $\alpha\in\overline{r+1,p}$, we obtain $A_{bc}^\alpha = 0$ and

$$\begin{array}{rcl} d^{h^*F}\Theta^\alpha & = & \Sigma_{b,\gamma}B_{b\gamma}^\alpha\Theta^b\wedge\Theta^\gamma + \frac{1}{2}C_{\beta\gamma}^\alpha\Theta^\beta\wedge\Theta^\gamma \\ & = & \left(B_{b\gamma}^\alpha\Theta^b + \frac{1}{2}C_{\beta\gamma}^\alpha\Theta^\beta\right)\wedge\Theta^\gamma. \end{array}$$

As

$$\Omega_\gamma^\alpha \overset{put}{=} B_{b\gamma}^\alpha\Theta^b + \frac{1}{2}C_{\beta\gamma}^\alpha\Theta^\beta \in \Lambda^1(h^*F, h^*\nu, M),$$

for any $\alpha,\beta\in\overline{r+1,p}$, it results the first implication.

Conversely, we admit that it exists

$$\Omega_\beta^\alpha\in\Lambda^1(h^*F, h^*\nu, M),\ \alpha,\beta\in\overline{r+1,p}$$

such that

$$d^{h^*F}\Theta^\alpha = \Sigma_{\beta\in\overline{r+1,p}}\Omega_\beta^\alpha\wedge\Theta^\beta, \tag{4}$$

for any $\alpha\in\overline{r+1,p}$.

Using the affirmations $(1), (2)$ and (4) we obtain that $A_{bc}^\alpha = 0$, for any $b,c\in\overline{1,r}$ and $\alpha\in\overline{r+1,p}$.

Using the affirmation (3), we obtain $\Theta^\alpha\left([S_b, S_c]_{h^*F}\right) = 0$, for any $b,c\in\overline{1,r}$ and $\alpha\in\overline{r+1,p}$.

Therefore, we have $[S_b, S_c]_{h^*F}\in\Gamma(E,\pi,M)$, for any $b,c\in\overline{1,r}$.

Using the *Proposition 3.2.2*, we obtain the second implication. *q.e.d.*

If $\left((F',\nu',N'), [,]_{F',h'}, (\rho',\eta')\right)$ is an another generalized Lie algebroid and (φ,φ_0) is a **GLA**-morphism of

$$\left((F,\nu,N), [,]_{F,h}, (\rho,\eta)\right)$$

source and

$$\left(\left(F', \nu', N' \right), [,]_{F',h'}, \left(\rho', \eta' \right) \right)$$

target, then obtain the application

$$\begin{array}{ccc} \Lambda^q \left(F', \nu', N' \right) & \xrightarrow{(\varphi,\varphi_0)^*} & \Lambda^q \left(F, \nu, N \right) \\ \omega' & \longmapsto & (\varphi, \varphi_0)^* \, \omega' \end{array},$$

where

$$\left((\varphi, \varphi_0)^* \, \omega' \right) (z_1, ..., z_q) = \omega' \left(\Gamma (\varphi, \varphi_0) (z_1), ..., \Gamma (\varphi, \varphi_0) (z_q) \right),$$

for any $z_1, ..., z_q \in \Gamma (F, \nu, N)$.

Theorem 4.8 *If* (φ, φ_0) *is a* **GLA**-*morphism of*

$$\left((F, \nu, N), [,]_{F,h}, (\rho, \eta) \right)$$

source and

$$\left(\left(F', \nu', N' \right), [,]_{F',h'}, \left(\rho', \eta' \right) \right)$$

target, then the following affirmations are satisfied:

1. *For any* $\omega' \in \Lambda^q (F', \nu', N')$ *and* $\theta' \in \Lambda^r (F', \nu', N')$ *we obtain*

$$(\varphi, \varphi_0)^* \left(\omega' \wedge \theta' \right) = (\varphi, \varphi_0)^* \, \omega' \wedge (\varphi, \varphi_0)^* \, \theta'. \tag{4.13}$$

2. *For any* $z \in \Gamma (F, \nu, N)$ *and* $\omega' \in \Lambda^q (F', \nu', N')$ *we obtain*

$$i_z \left((\varphi, \varphi_0)^* \, \omega' \right) = (\varphi, \varphi_0)^* \left(i_{\varphi(z)} \omega' \right). \tag{4.14}$$

3. *If* $N = N'$ *and* $(Th \circ \rho, h \circ \eta) = (Th' \circ \rho', h' \circ \eta') \circ (\varphi, \varphi_0)$, *then we obtain*

$$(\varphi, \varphi_0)^* \circ d^{F'} = d^F \circ (\varphi, \varphi_0)^*. \tag{4.15}$$

4.1 Exterior Differential Systems

Let $\left((h^* F, h^* \nu, M), [,]_{h^* F}, \left(\overset{h^* F}{\rho}, Id_M \right) \right)$ be the pull-back Lie algebroid of the generalized Lie algebroid $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta) \right)$.

Definition 4.1.1 Any ideal $(\mathcal{I}, +, \cdot)$ of the exterior differential algebra of the pull-back Lie algebroid $\left((h^* F, h^* \nu, M), [,]_{h^* F}, \left(\overset{h^* F}{\rho}, Id_M \right) \right)$ closed under differentiation operator $d^{h^* F}$, namely $d^{h^* F} \mathcal{I} \subseteq \mathcal{I}$, will be called *differential ideal of the generalized Lie algebroid* $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta) \right)$.

In particular, if $h = Id_N = \eta$, then any ideal $(\mathcal{I}, +, \cdot)$ of the exterior differential algebra of the Lie algebroid $((F, \nu, N), [,]_F, (\rho, Id_M))$ closed under differentiation operator d^F, namely $d^F \mathcal{I} \subseteq \mathcal{I}$, will be called *differential ideal of the Lie algebroid* $((F, \nu, N), [,]_F, (\rho, Id_M))$.

Definition 4.1.2 Let $(\mathcal{I}, +, \cdot)$ be a differential ideal of the generalized Lie algebroid $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta) \right)$ or of the Lie algebroid $((F, \nu, N), [,]_F, (\rho, Id_M))$ respectively.

If it exists an IDS (E,π,M) such that for all $k\in\mathbb{N}^*$ and $\omega\in\mathcal{I}\cap\Lambda^k(h^*F,h^*\nu,M)$ we have $\omega(u_1,...,u_k)=0$, for any $u_1,...,u_k\in\Gamma(E,\pi,M)$, then we will say that $(\mathcal{I},+,\cdot)$ *is an exterior differential system (EDS) of the generalized Lie algebroid*

$$\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right).$$

In particular, if $h=Id_N=\eta$ and it exists an IDS (E,π,M) such that for all $k\in\mathbb{N}^*$ and $\omega\in\mathcal{I}\cap\Lambda^k(F,\nu,M)$ we have $\omega(u_1,...,u_k)=0$, for any $u_1,...,u_k\in\Gamma(E,\pi,M)$, then we will say that $(\mathcal{I},+,\cdot)$ *is an exterior differential system (EDS) of the Lie algebroid* $((F,\nu,N),[,]_F,(\rho,Id_N))$.

Theorem 4.1.1 (of Cartan type) *The IDS (E,π,M) of the generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is involutive, if and only if the ideal generated by the $\mathcal{F}(M)$-submodule $\left(\Gamma\left(E^0,\pi^0,M\right),+,\cdot\right)$ is an EDS of the generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$.*

Proof. Let (E,π,M) be an involutive IDS of the generalized Lie algebroid

$$\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right).$$

Let $\left\{\Theta^{r+1},...,\Theta^p\right\}$ be a base for the $\mathcal{F}(M)$-submodule $\left(\Gamma\left(E^0,\pi^0,M\right),+,\cdot\right)$.

We know that

$$\mathcal{I}\left(\Gamma\left(E^0,\pi^0,M\right)\right)=\cup_{q\in\mathbb{N}}\left\{\Omega_\alpha\wedge\Theta^\alpha,\ \ \{\Omega_{r+1},...,\Omega_p\}\subset\Lambda^q(h^*F,h^*\nu,M)\right\}.$$

Let $q\in\mathbb{N}$ and $\{\Omega_{r+1},...,\Omega_p\}\subset\Lambda^q(h^*F,h^*\nu,M)$ be arbitrary.

Using the *Theorems 4.5 and 4.7* we obtain

$$\begin{array}{rcl} d^{h^*F}(\Omega_\alpha\wedge\Theta^\alpha) & = & d^{h^*F}\Omega_\alpha\wedge\Theta^\alpha+(-1)^{q+1}\Omega_\beta\wedge d^{h^*F}\Theta^\beta\\ & = & \left(d^{h^*F}\Omega_\alpha+(-1)^{q+1}\Omega_\beta\wedge\Omega_\alpha^\beta\right)\wedge\Theta^\alpha. \end{array}$$

As

$$d^{h^*F}\Omega_\alpha+(-1)^{q+1}\Omega_\beta\wedge\Omega_\alpha^\beta\in\Lambda^{q+2}(h^*F,h^*\nu,M)$$

it results that

$$d^{h^*F}\left(\Omega_\beta\wedge\Theta^\beta\right)\in\mathcal{I}\left(\Gamma\left(E^0,\pi^0,M\right)\right)$$

Therefore,

$$d^{h^*F}\mathcal{I}\left(\Gamma\left(E^0,\pi^0,M\right)\right)\subseteq\mathcal{I}\left(\Gamma\left(E^0,\pi^0,M\right)\right).$$

Conversely, let (E,π,M) be an IDS of the generalized Lie algebroid

$$\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$$

such that the $\mathcal{F}(M)$-submodule $\left(\mathcal{I}\left(\Gamma\left(E^0,\pi^0,M\right)\right),+,\cdot\right)$ is an EDS of the generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$.

Let $\left\{\Theta^{r+1},...,\Theta^p\right\}$ be a base for the $\mathcal{F}(M)$-submodule $\left(\Gamma\left(E^0,\pi^0,M\right),+,\cdot\right)$. As

$$d^{h^*F}\mathcal{I}\left(\Gamma\left(E^0,\pi^0,M\right)\right)\subseteq\mathcal{I}\left(\Gamma\left(E^0,\pi^0,M\right)\right)$$

it results that it exists

$$\Omega_\beta^\alpha\in\Lambda^1(h^*F,h^*\nu,M),\ \alpha,\beta\in\overline{r+1,p}$$

such that

$$d^{h^*F}\Theta^\alpha=\Sigma_{\beta\in\overline{r+1,p}}\Omega_\beta^\alpha\wedge\Theta^\beta\in\mathcal{I}\left(\Gamma\left(E^0,\pi^0,M\right)\right).$$

Using the *Theorem 4.7*, it results that (E,π,M) is an involutive IDS. *q.e.d.*

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ALGEBRAIC CONSTRUCTIONS IN THE CATEGORY OF LIE ALGEBROIDS

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Abstract

A generalized notion of a Lie algebroid is presented. Using this, the Lie algebroid generalized tangent bundle is obtained. A new point of view over (linear) connections theory on a fiber bundle is presented. These connections are characterized by o horizontal distribution of the Lie algebroid generalized tangent bundle. Some basic properties of these generalized connections are investigated. Special attention to the class of linear connections is paid. The recently studied Lie algebroids connections can be recovered as special cases within this more general framework. In particular, all results are similar with the classical results. Formulas of Ricci type and linear connections of Levi-Civita type are presented.

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Contents

1 Introduction

The theory of connections constitutes one of the most important chapter of differential geometry, which has been explored in the literature (see [2, 3, 4, 5, 10, 11, 12, 13, 14, 15, 16]). Connections theory has become an indispensable tool in various branches of theoretical and mathematical physics.

If (E, π, M) is a fiber bundle with paracompact base and (VTE, τ_E, E) is the kernel vector bundle of the tangent $\mathbf{B^v}$-morphism $(T\pi, \pi)$, then we obtain the short exact sequence

$$(1)\qquad \begin{array}{ccccccccc} 0 & \hookrightarrow & VTE & \hookrightarrow & TE & \xrightarrow{\pi!} & \pi^*TM & \longrightarrow & 0 \\ & & \downarrow \tau_E & & \downarrow \tau_E & & \downarrow \pi^*\tau_M & & \\ & & E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E & & \end{array}$$

where $\pi!$ is the projection of TE onto π^*TM.

We know that a split to the right in the previous short exact sequence, i.e. a smooth map $h \in \mathbf{Man}(\pi^*TM, TE)$ so that $\pi! \circ h = Id_{\pi^*TM}$, is a *connection in the Ehresmann sense.*

If (HTE, τ_E, E) is the image vector bundle of the $\mathbf{B^v}$-morphism (h, Id_E), then the tangent vector bundle (TE, τ_E, E) is a Whitney sum between the *horizontal vector bundle* (HTE, τ_E, E) and the *vertical vector bundle* (VTE, τ_E, E).

From the above notion of connection, one can easily derive more specific types of connections by imposing additional conditions. In the literature one can find several generalizations of the concept of Ehresmann connection obtained by relaxing the conditions on the horizontal vector bundle.

- First of all, we are thinking here of the so-called *partial connections,* where (HTE, τ_E, E) does not determine a full complement of (VTE, τ_E, E). More precisely, $\Gamma(HTE, \tau_E, E)$ has zero intersection with $\Gamma(VTE, \tau_E, E)$, but (HTE, τ_E, E) projects onto a vector subbundle of (TM, τ_M, M). (see [7])

- Secondly, there also exists a notion of *pseudo-connection*, introduced under the name of *quasi-connection* in a paper by Y. C. Wong [16]. Linear pseudo-connections and generalization of it have been studied by many authors. (see [3])

P. Popescu build the *relativ tangent space* and using that he obtained a new *generalized connection* on a vector bundle.[11] (see also [12])

In the paper [4] by R. L. Fernandez a *contravariant connection* in the framework of Poisson Geometry there are presented. Given a Poisson manifold M with tensor Λ which does not have to be of constant rank, a *covariant connection* on the principal bundle (P, π, M) is a G-invariant bundle map $h \in \mathbf{Man}(\pi^*(T^*M), TP)$ so that the diagram is commutative

$$(2)\qquad \begin{array}{ccc} \pi^*(T^*M) & \xrightarrow{h} & TP \\ \pi^*\left(\overset{*}{\tau}_M\right)\downarrow & & \downarrow T\pi \\ T^*M & \xrightarrow{\sharp_\Lambda} & TM \end{array}$$

where $(\sharp_\Lambda, Id_M)$ is the natural vector bundle morphism induced by the Poisson tensor. In the paper [5], R. L. Fernandez has extending this theory by replacing the cotangent

bundle of a Poisson manifold by a Lie algebroid over an arbitrary manifold and the map $\sharp_\Lambda$ by the anchor map of the Lie algebroid. This resulted into a notion of *Lie algebroid connection* which, in particular, turns out to be appropriate for studying the geometry of singular distributions.

B. Langerock and F. Cantrijn [2] proposed a *general notion of connection* on a fiber bundle (E,π,M), as being a smooth linear bundle map $h\in\mathbf{Man}(\pi^*(F),TE)$ so that the diagram is commutative

$$\begin{array}{ccc} \pi^*(F) & \stackrel{h}{\longrightarrow} & TE \\ \downarrow & & \downarrow T\pi \\ F & \stackrel{\rho}{\longrightarrow} & TM \end{array} \tag{3}$$

where (F,ν,M) is an arbitrary vector bundle and (ρ, Id_M) is a vector bundle morphism of (F,ν,M) source and (TM,τ_M,M) target.

Different equivalent definitions of a (linear) connection on a vector bundle are known and there are in current usage. We know the following

Theorem *If we have a short exact sequence of vector bundles over a paracompact manifold M*

$$\begin{array}{ccccccc} 0 & \hookrightarrow & E' & \stackrel{f}{\longrightarrow} & E & \stackrel{g}{\longrightarrow} & E'' & \longrightarrow & 0 \\ & & \downarrow\pi' & & \downarrow\pi & & \downarrow\pi'' & & \\ & & M & \stackrel{Id_M}{\longrightarrow} & M & \stackrel{Id_M}{\longrightarrow} & M & & \end{array} \tag{4}$$

then there exists a right split if and only if there exists a left split.

So, a split to the left in the short exact sequence (1), i.e. a smooth map $\Gamma\in\mathbf{Man}(TE,VTE)$ so that $\Gamma\circ i=Id_{TE}$, is an equivalent definition with the Ehresmann connection.

We remark that the secret of the Ehresmann connection is given by the diagrams

$$\begin{array}{ccccc} E & & (TM,[,]_{TM}) & \stackrel{Id_{TM}}{\longrightarrow} & (TM,[,]_{TM}) \\ \downarrow\pi & & \downarrow\tau_M & & \downarrow\tau_M \\ M & \stackrel{Id_M}{\longrightarrow} & M & \stackrel{Id_M}{\longrightarrow} & M \end{array} \tag{5}$$

where (E,π,M) is a fiber bundle and $((TM,\tau_M,M),[,]_{TM},(Id_{TM},Id_M))$ is the standard Lie algebroid.

First time, appeared the idea to change the standard Lie algebroid with an arbitrary Lie algebroid as in the diagrams

$$\begin{array}{ccccc} E & & (F,[,]_F) & \stackrel{\rho}{\longrightarrow} & (TM,[,]_{TM}) \\ \downarrow\pi & & \downarrow\nu & & \downarrow\tau_M \\ M & \stackrel{Id_M}{\longrightarrow} & M & \stackrel{Id_M}{\longrightarrow} & M \end{array} \tag{6}$$

Second time, appeared the idea to change in the previous diagrams the identities morphisms with arbitrary **Man**-isomorphisms h and η as in the diagrams

$$\begin{array}{ccccccc} E & & \left(F,[,]_{F,h}\right) & \stackrel{\rho}{\longrightarrow} & (TM,[,]_{TM}) & \stackrel{Th}{\longrightarrow} & (TN,[,]_{TN}) \\ \downarrow\pi & & \downarrow\nu & & \downarrow\tau_M & & \downarrow\tau_N \\ M & \stackrel{h}{\longrightarrow} & N & \stackrel{\eta}{\longrightarrow} & M & \stackrel{h}{\longrightarrow} & N \end{array} \tag{7}$$

where

$$(\rho, \eta) \in \mathbf{B^v}\left(\left(F, \nu, M\right), \left(TM, \tau_M, M\right)\right)$$

and

$$\begin{array}{ccc} \Gamma\left(F, \nu, N\right) \times \Gamma\left(F, \nu, N\right) & \xrightarrow{[,]_{F,h}} & \Gamma\left(F, \nu, N\right) \\ (u, v) & \longmapsto & [u, v]_{F,h} \end{array}$$

is an operation with the following properties:

GLA_1. *the equality holds good*

$$[u, f \cdot v]_{F,h} = f\,[u, v]_{F,h} + \Gamma\left(Th \circ \rho, h \circ \eta\right)(u)\, f \cdot v,$$

for all $u, v \in \Gamma\left(F, \nu, N\right)$ *and* $f \in \mathcal{F}(N)$.

GLA_2. *the* 4*-tuple* $\left(\Gamma\left(F, \nu, N\right), +, \cdot, [,]_{F,h}\right)$ *is a Lie* $\mathcal{F}(N)$*-algebra,*

GLA_3. *the* **Mod***-morphism* $\Gamma\left(Th \circ \rho, h \circ \eta\right)$ *is a* **LieAlg***-morphism of* $\left(\Gamma\left(F, \nu, N\right), +, \cdot, [,]_{F,h}\right)$ *source and* $\left(\Gamma\left(TN, \tau_N, N\right), +, \cdot, [,]_{TN}\right)$ *target.*

So, appeared the notion of *generalized Lie algebroid* which is presented in *Definition 2.1.* Using this new notion we build the *Lie algebroid generalized tangent bundle* in the *Theorem 3.1.* Particularly, if $\left(\left(F, \nu, N\right), [,]_F, \left(\rho, Id_N\right)\right)$ is a Lie algebroid, $(E, \pi, M) = (F, \nu, N)$ and $h = Id_M$, then we obtain a similar Lie algebroid with the the *prolongation Lie algebroid*. (see [6, 8, 9, 10]) Using this general framework, in Section 4, we propose and develop a (linear) connections theory of Ehresmann type for fiber bundles in general and for vector bundles in particular. It covers all types of connections mentioned. In this general framework, we can define the covariant derivatives of sections of a fiber bundle (E, π, M) with respect to sections of a generalized Lie algebroid

$$\left(\left(F, \nu, N\right), [,]_{F,h}, (\rho, \eta)\right).$$

In particular, if we use the generalized Lie algebroid structure

$$\left([,]_{TM, Id_M}, \left(Id_{TM}, Id_M\right)\right)$$

for the tangent bundle (TM, τ_M, M) in our theory, then the linear connections obtained are similar with the classical linear connections.

It is known that in Yang-Mills theory the set

$$Cov^0_{(E,\pi,M)}$$

of covariant derivatives for the vector bundle (E, π, M) such that

$$X\left(\langle u, v\rangle_E\right) = \langle D_X(u), v\rangle_E + \langle u, D_X(v)\rangle_E,$$

for any $X \in \mathcal{X}(M)$ and $u, v \in \Gamma(E, \pi, M)$, is very important, because the Yang-Mills theory is a variational theory which use (see [1]) the Yang-Mills functional

$$\begin{array}{ccc} Cov^0_{(E,\pi,M)} & \xrightarrow{\mathcal{YM}} & \mathbb{R} \\ D_X & \longmapsto & \frac{1}{2}\int\limits_M \left\|\mathbb{R}^{D_X}\right\|^2 v_g \end{array}$$

where $\mathbb{R}^{D_X}$ is the curvature.

Using our linear connections theory, we succeed to extend the set $Cov^0_{(E,\pi,M)}$ of Yang-Mills theory, because using all generalized Lie algebroid structures for the tangent bundle (TM,τ_M,M), we obtain all possible linear connections for the vector bundle (E,π,M).

More importantly, it may bring within the reach of connection theory certain geometric structures which have not yet been considered from such a point of view. Finally, using our theory of linear connections, the formulas of Ricci and Bianchi type and linear connections of Levi-Civita type are presented.

2 Preliminaries

In general, if $\mathcal{C}$ is a category, then we denote $|\mathcal{C}|$ the class of objects and for any $A, B \in |\mathcal{C}|$, we denote $\mathcal{C}(A,B)$ the set of morphisms of A source and B target. Let **Vect**, **Liealg**, **Mod, Man**, **B** and $\mathbf{B^v}$ be the category of real vector spaces, Lie algebras, modules, manifolds, fiber bundles and vector bundles respectively.

We know that if $(E,\pi,M) \in |\mathbf{B^v}|$, $\Gamma(E,\pi,M) = \{u \in \mathbf{Man}(M,E) : u \circ \pi = Id_M\}$ and $\mathcal{F}(M) = \mathbf{Man}(M,\mathbb{R})$, then $(\Gamma(E,\pi,M),+,\cdot)$ is a $\mathcal{F}(M)$-module. If $(\varphi,\varphi_0) \in \mathbf{B^v}((E,\pi,M),(E',\pi',M'))$ such that $\varphi_0 \in Iso_{\mathbf{Man}}(M,M')$, then, using the operation

$$\begin{array}{ccc} \mathcal{F}(M) \times \Gamma(E',\pi',M') & \xrightarrow{\cdot} & \Gamma(E',\pi',M') \\ (f,u') & \longmapsto & f \circ \varphi_0^{-1} \cdot u \end{array}$$

it results that $(\Gamma(E',\pi',M'),+,\cdot)$ is a $\mathcal{F}(M)$-module and we obtain the **Mod**-morphism

$$\begin{array}{ccc} \Gamma(E,\pi,M) & \xrightarrow{\Gamma(\varphi,\varphi_0)} & \Gamma(E',\pi',M') \\ u & \longmapsto & \Gamma(\varphi,\varphi_0)\,u \end{array}$$

defined by

$$\Gamma(\varphi,\varphi_0)\,u(y) = \varphi\left(u_{\varphi_0^{-1}(y)}\right),$$

for any $y \in M'$.

We know that a Lie algebroid is a vector bundle $(F,\nu,N) \in |\mathbf{B^v}|$ such that there exists

$$(\rho, Id_N) \in \mathbf{B^v}((F,\nu,N),(TN,\tau_N,N))$$

and an operation

$$\begin{array}{ccc} \Gamma(F,\nu,N) \times \Gamma(F,\nu,N) & \xrightarrow{[,]_F} & \Gamma(F,\nu,N) \\ (u,v) & \longmapsto & [u,v]_F \end{array}$$

with the following properties:

LA_1. the equality holds good

$$[u, f\cdot v]_F = f[u,v]_F + \Gamma(\rho, Id_N)(u) f \cdot v,$$

for all $u,v \in \Gamma(F,\nu,N)$ and $f \in \mathcal{F}(N)$,

LA_2. the 4-tuple $(\Gamma(F,\nu,N),+,\cdot,[,]_F)$ is a Lie $\mathcal{F}(N)$-algebra,

LA_3. the **Mod**-morphism $\Gamma(\rho, Id_N)$ is a **LieAlg**-morphism of $(\Gamma(F,\nu,N),+,\cdot,[,]_F)$ source and $(\Gamma(TN,\tau_N,N),+,\cdot,[,]_{TN})$ target.

Definition 2.1 Let $M, N \in |\mathbf{Man}|$, $h \in Iso_{\mathbf{Man}}(M, N)$ and $\eta \in Iso_{\mathbf{Man}}(N, M)$. If $(F, \nu, N) \in |\mathbf{B^v}|$ so that there exists

$$(\rho, \eta) \in \mathbf{B^v}((F, \nu, N), (TM, \tau_M, M))$$

and an operation

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[,]_{F,h}} & \Gamma(F, \nu, N) \\ (u, v) & \longmapsto & [u, v]_{F,h} \end{array}$$

with the following properties:

GLA_1. the equality holds good

$$[u, f \cdot v]_{F,h} = f[u, v]_{F,h} + \Gamma(Th \circ \rho, h \circ \eta)(u) f \cdot v,$$

for all $u, v \in \Gamma(F, \nu, N)$ and $f \in \mathcal{F}(N)$.

GLA_2. the 4-tuple $\left(\Gamma(F, \nu, N), +, \cdot, [,]_{F,h}\right)$ is a Lie $\mathcal{F}(N)$-algebra,

GLA_3. the **Mod**-morphism $\Gamma(Th \circ \rho, h \circ \eta)$ is a **LieAlg**-morphism of $\left(\Gamma(F, \nu, N), +, \cdot, [,]_{F,h}\right)$ source and $(\Gamma(TN, \tau_N, N), +, \cdot, [,]_{TN})$ target, then we will say that *the triple*

$$\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right) \tag{2.1}$$

is a generalized Lie algebroid. The couple $\left([,]_{F,h}, (\rho, \eta)\right)$ will be called *generalized Lie algebroid structure.*

Let $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$ be a generalized Lie algebroid.

- Locally, for any $\alpha, \beta \in \overline{1, p}$, we set $[t_\alpha, t_\beta]_{F,h} = L_{\alpha\beta}^\gamma t_\gamma$. We easily obtain that $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$, for any $\alpha, \beta, \gamma \in \overline{1, p}$.

The real local functions $L_{\alpha\beta}^\gamma$, $\alpha, \beta, \gamma \in \overline{1, p}$ will be called the *structure functions of the generalized Lie algebroid* $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$.

- We assume the following diagrams:

$$\begin{array}{ccccc} F & \xrightarrow{\rho} & TM & \xrightarrow{Th} & TN \\ \downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\ N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \\ & & & & \\ (\chi^{\tilde{i}}, z^\alpha) & & (x^i, y^i) & & (\chi^{\tilde{i}}, z^{\tilde{i}}) \end{array}$$

where $i, \tilde{i} \in \overline{1, m}$ and $\alpha \in \overline{1, p}$.

If

$$(\chi^{\tilde{i}}, z^\alpha) \longrightarrow (\chi^{\tilde{i}\prime}(\chi^{\tilde{i}}), z^{\alpha\prime}(\chi^{\tilde{i}}, z^\alpha)),$$

$$(x^i, y^i) \longrightarrow \left(x^{i\prime}(x^i), y^{i\prime}(x^i, y^i)\right)$$

and

$$\left(\chi^{\tilde{\imath}}, z^{\tilde{\imath}}\right) \longrightarrow \left(\chi^{\tilde{\imath}\prime}\left(\chi^{\tilde{\imath}}\right), z^{\tilde{\imath}\prime}\left(\chi^{\tilde{\imath}}, z^{\tilde{\imath}}\right)\right),$$

then

$$z^{\acute{\alpha}} = \Lambda_{\alpha}^{\acute{\alpha}} z^{\alpha} \quad ,$$

$$y^{\acute{\imath}} = \tfrac{\partial x^{\acute{\imath}}}{\partial x^{i}} y^{i}$$

and

$$z^{\tilde{\imath}\prime} = \tfrac{\partial \chi^{\tilde{\imath}\prime}}{\partial \chi^{\tilde{\imath}}} z^{\tilde{\imath}}.$$

- We assume that $(\theta, \mu) \overset{put}{=} (Th \circ \rho, h \circ \eta)$. If $z^{\alpha} t_{\alpha} \in \Gamma(F, \nu, N)$ is arbitrary, then

$$\begin{array}{c} \Gamma\left(Th \circ \rho, h \circ \eta\right)\left(z^{\alpha} t_{\alpha}\right) f\left(h \circ \eta\left(\varkappa\right)\right) = \\ = \left(\theta_{\alpha}^{\tilde{\imath}} z^{\alpha} \tfrac{\partial f}{\partial \varkappa^{\tilde{\imath}}}\right)\left(h \circ \eta\left(\varkappa\right)\right) = \left(\left(\rho_{\alpha}^{i} \circ h\right)\left(z^{\alpha} \circ h\right) \tfrac{\partial f \circ h}{\partial x^{i}}\right)\left(\eta\left(\varkappa\right)\right), \end{array} \tag{2.2}$$

for any $f \in \mathcal{F}(N)$ and $\varkappa \in N$.

The coefficients ρ_{α}^{i} respectively $\theta_{\alpha}^{\tilde{\imath}}$ change to $\rho_{\acute{\alpha}}^{\acute{\imath}}$ respectively $\theta_{\acute{\alpha}}^{\tilde{\imath}}$ according to the rule:

$$\rho_{\acute{\alpha}}^{\acute{\imath}} = \Lambda_{\acute{\alpha}}^{\alpha} \rho_{\alpha}^{i} \frac{\partial x^{\acute{\imath}}}{\partial x^{i}}, \tag{2.3}$$

respectively

$$\theta_{\acute{\alpha}}^{\tilde{\imath}} = \Lambda_{\acute{\alpha}}^{\alpha} \theta_{\alpha}^{\tilde{\imath}} \frac{\partial \varkappa^{\tilde{\imath}}}{\partial \varkappa^{\tilde{\imath}}}, \tag{2.4}$$

where

$$\left\|\Lambda_{\acute{\alpha}}^{\alpha}\right\| = \left\|\Lambda_{\alpha}^{\acute{\alpha}}\right\|^{-1}.$$

Remark 2.2 The following equalities hold good:

$$\rho_{\alpha}^{i} \circ h \frac{\partial f \circ h}{\partial x^{i}} = \left(\theta_{\alpha}^{\tilde{\imath}} \frac{\partial f}{\partial \varkappa^{\tilde{\imath}}}\right) \circ h, \forall f \in \mathcal{F}(N). \tag{2.5}$$

and

$$\left(L_{\alpha\beta}^{\gamma} \circ h\right)\left(\rho_{\gamma}^{k} \circ h\right) = \left(\rho_{\alpha}^{i} \circ h\right) \frac{\partial\left(\rho_{\beta}^{k} \circ h\right)}{\partial x^{i}} - \left(\rho_{\beta}^{j} \circ h\right) \frac{\partial\left(\rho_{\alpha}^{k} \circ h\right)}{\partial x^{j}}. \tag{2.6}$$

Theorem 2.1 *Let $M, N \in |\mathbf{Man}|$, $h \in Iso_{\mathbf{Man}}(M, N)$ and $\eta \in Iso_{\mathbf{Man}}(N, M)$ be. Using the tangent $\mathbf{B^v}$-morphism $(T\eta, \eta)$ and the operation*

$$\begin{array}{ccc} \Gamma\left(TN, \tau_N, N\right) \times \Gamma\left(TN, \tau_N, N\right) & \xrightarrow{[,]_{TN,h}} & \Gamma\left(TN, \tau_N, N\right) \\ (u, v) & \longmapsto & [u, v]_{TN,h} \end{array}$$

where

$$[u, v]_{TN,h} = \Gamma\left(T\left(h \circ \eta\right)^{-1}, \left(h \circ \eta\right)^{-1}\right)\left(\left[\Gamma\left(T\left(h \circ \eta\right), h \circ \eta\right) u, \Gamma\left(T\left(h \circ \eta\right), h \circ \eta\right) v\right]_{TN}\right),$$

for any $u, v \in \Gamma(TN, \tau_N, N)$*, we obtain that*

$$\left(\left(TN, \tau_N, N\right), \left(T\eta, \eta\right), [,]_{TN,h}\right)$$

is a generalized Lie algebroid.

For any **Man**-isomorphisms η and h, new and interesting generalized Lie algebroid structures for the tangent vector bundle (TN, τ_N, N) are obtained. For any base $\left\{t_\alpha,\ \alpha \in \overline{1,m}\right\}$ of the module of sections $\left(\Gamma\left(TN, \tau_N, N\right), +, \cdot\right)$ we obtain the structure functions

$$L_{\alpha\beta}^{\gamma} = \left(\theta_\alpha^i \frac{\partial \theta_\beta^j}{\partial x^i} - \theta_\beta^i \frac{\partial \theta_\alpha^j}{\partial x^i}\right) \tilde{\theta}_j^\gamma,\ \ \alpha, \beta, \gamma \in \overline{1,m}$$

where

$$\theta_\alpha^i,\ \ i, \alpha \in \overline{1,m}$$

are real local functions so that

$$\Gamma\left(T\left(h \circ \eta\right), h \circ \eta\right)\left(t_\alpha\right) = \theta_\alpha^i \frac{\partial}{\partial x^i}$$

and

$$\tilde{\theta}_j^\gamma,\ \ i, \gamma \in \overline{1,m}$$

are real local functions so that

$$\Gamma\left(T\left(h \circ \eta\right)^{-1}, \left(h \circ \eta\right)^{-1}\right)\left(\frac{\partial}{\partial x^j}\right) = \tilde{\theta}_j^\gamma t_\gamma.$$

In particular, using arbitrary isometries (symmetries, translations, rotations,...) for the Euclidean 3-dimensional space Σ, and arbitrary basis for the module of sections we obtain a lot of generalized Lie algebroid structures for the tangent vector bundle $(T\Sigma, \tau_\Sigma, \Sigma)$.

Let $\left(\left(F, \nu, M\right), [,]_F, \left(\rho, Id_M\right)\right)$ be a Lie algebroid and let $h \in Iso_{\mathbf{Man}}\left(M, M\right)$ be. Let $\mathcal{AF}_F$ be a vector fibred $(m+p)$-atlas for the vector bundle (F, ν, M) and let $\mathcal{AF}_{TM}$ be a vector fibred $(m+m)$-atlas for the vector bundle (TM, τ_M, M).

If $(U, \xi_U) \in \mathcal{AF}_{TM}$ and $(V, s_V) \in \mathcal{AF}_F$ such that $U \cap h^{-1}(V) \neq \phi$, then we define the application

$$\begin{array}{ccc} \tau_N^{-1}(U \cap h^{-1}(V))) & \xrightarrow{\bar{\xi}_{U \cap h^{-1}(V)}} & (U \cap h^{-1}(V)) \times \mathbb{R}^m \\ (\varkappa, u(\varkappa)) & \longmapsto & \left(\varkappa, \xi_{U,\varkappa}^{-1} u(\varkappa)\right). \end{array}$$

Proposition 2.1 *The set*

$$\overline{\mathcal{AF}}_{TM} \overset{put}{=} \bigcup_{\substack{(U,\xi_U) \in \mathcal{AF}_{TM},\ (V,s_V) \in \mathcal{AF}_F \\ U \cap h^{-1}(V) \neq \phi}} \left\{\left(U \cap h^{-1}\left(V\right), \bar{\xi}_{U \cap h^{-1}(V)}\right)\right\}$$

is a vector fibred $m+m$*-atlas of the vector bundle* (TM, τ_M, M).

If $X = X^{\tilde{\imath}} \frac{\partial}{\partial \varkappa^{\tilde{\imath}}} \in \Gamma\left(TM, \tau_M, M\right)$, *then we obtain the section*

$$\bar{X} = \bar{X}^{\tilde{\imath}} \circ h \frac{\partial}{\partial \bar{\varkappa}^{\tilde{\imath}}} \in \Gamma\left(TM, \tau_M, M\right),$$

such that $\bar{X}\left(\bar{\varkappa}\right) = X\left(h\left(\bar{\varkappa}\right)\right)$, *for any* $\bar{\varkappa} \in U \cap h^{-1}\left(V\right)$.

The set $\left\{\frac{\partial}{\partial \bar{\varkappa}^{\tilde{\imath}}},\ \tilde{\imath} \in \overline{1,m}\right\}$ is the natural base of the $\mathcal{F}(M)$-module $\left(\Gamma\left(TM, \tau_M, M\right), +, \cdot\right)$.

Theorem 2.2 *If we consider the operation*

$$\Gamma\left(F,\nu,M\right)\times\Gamma\left(F,\nu,M\right)\ \xrightarrow{[,]_{F,h}}\ \Gamma\left(F,\nu,M\right)$$

defined by

$$\begin{array}{ll}\left[t_{\alpha},t_{\beta}\right]_{F,h} & =\left(L_{\alpha\beta}^{\gamma}\circ h\right)t_{\gamma},\\ \left[t_{\alpha},ft_{\beta}\right]_{F,h} & =f\left(L_{\alpha\beta}^{\gamma}\circ h\right)t_{\gamma}+\rho_{\alpha}^{\tilde{\imath}}\circ h\dfrac{\partial f}{\partial\tilde{\varkappa}^{\tilde{\imath}}}t_{\beta},\\ \left[ft_{\alpha},t_{\beta}\right]_{F,h} & =-\left[t_{\beta},ft_{\alpha}\right]_{F,h},\end{array}$$

for any $f\in\mathcal{F}\left(M\right)$, *then* $\Big(\left(F,\nu,M\right),[,]_{F,h},\left(\rho,Id_{M}\right)\Big)$ *is a generalized Lie algebroid.*

This generalized Lie algebroid is called *the generalized Lie algebroid associated to the Lie algebroid* $\left(\left(F,\nu,M\right),[,]_{F},\left(\rho,Id_{M}\right)\right)$ *and to the diffeomorphism* h.

In particular, if $h=Id_{M}$, then the generalized Lie algebroid

$$\Big(\left(F,\nu,M\right),[,]_{F,Id_{M}},\left(\rho,Id_{M}\right)\Big)$$

will be called *the generalized Lie algebroid associated to the Lie algebroid*

$$\left(\left(F,\nu,M\right),[,]_{F},\left(\rho,Id_{M}\right)\right).$$

Let $\Big(\left(F,\nu,N\right),[,]_{F,h},\left(\rho,\eta\right)\Big)$ be a generalized Lie algebroid.

Let $\mathcal{AF}_{TM}$ be a vector fibred $(m+m)$-atlas for the vector bundle $\left(TM,\tau_{M},M\right)$ and let $\left(h^{*}F,h^{*}\nu,M\right)$ be the pull-back vector bundle through h. If $\left(U,\xi_{U}\right)\in\mathcal{AF}_{TM}$ and $\left(V,t_{V}\right)\in\mathcal{AF}_{F}$ so that $U\cap h^{-1}\left(V\right)\neq\phi$, then we define the application

$$\begin{array}{ccc}h^{*}\nu^{-1}(U\cap h^{-1}(V))) & \xrightarrow{\bar{s}_{U\cap h^{-1}(V)}} & \big(U\cap h^{-1}(V)\big)\times\mathbb{R}^{p}\\ \left(\varkappa,z\left(h\left(\varkappa\right)\right)\right) & \longmapsto & \left(\varkappa,t_{V,h(\varkappa)}^{-1}z\left(h\left(\varkappa\right)\right)\right).\end{array}$$

Proposition 2.2 *The set*

$$\overline{\mathcal{AF}}_{F}\overset{put}{=}\bigcup_{\substack{(U,\xi_{U})\in\mathcal{AF}_{TM},\ (V,s_{V})\in\mathcal{AF}_{F}\\ U\cap h^{-1}(V)\neq\phi}}\left\{\left(U\cap h^{-1}\left(V\right),\bar{s}_{U\cap h^{-1}(V)}\right)\right\}$$

is a vector fibred $m+p$*-atlas for the vector bundle* $\left(h^{*}F,h^{*}\nu,M\right)$.

If $z=z^{\alpha}t_{\alpha}\in\Gamma\left(F,\nu,N\right)$, *then we obtain the section*

$$Z=\left(z^{\alpha}\circ h\right)T_{\alpha}\in\Gamma\left(h^{*}F,h^{*}\nu,M\right)$$

so that $Z\left(x\right)=z\left(h\left(x\right)\right)$, *for any* $x\in U\cap h^{-1}\left(V\right)$.

In addition, we obtain the inclusion $\mathbf{B^{v}}$-morphism

$$\begin{array}{rcl}h^{*}F & \hookrightarrow & F\\ h^{*}\nu\downarrow & & \downarrow\nu\\ M & \xrightarrow{h} & M\end{array}\tag{2.7}$$

Theorem 2.3 *Let* $\left(\overset{h^*F}{\rho}, Id_M\right)$ *be the* $\mathbf{B^v}$*-morphism of* $(h^*F, h^*\nu, M)$ *source and* (TM, τ_M, M) *target, where*

$$
(2.8) \qquad \begin{array}{rcl} h^*F & \overset{\overset{h^*F}{\rho}}{\longrightarrow} & TM \\ Z^\alpha T_\alpha(x) & \longmapsto & \left(Z^\alpha \cdot \rho_\alpha^i \circ h\right) \dfrac{\partial}{\partial x^i}(x) \end{array}
$$

Using the operation

$$
\Gamma(h^*F, h^*\nu, M) \times \Gamma(h^*F, h^*\nu, M) \xrightarrow{[,]_{h^*F}} \Gamma(h^*F, h^*\nu, M)
$$

defined by

$$
(2.9) \qquad \begin{array}{ll} [T_\alpha, T_\beta]_{h^*F} & = \left(L_{\alpha\beta}^\gamma \circ h\right) T_\gamma, \\ [T_\alpha, fT_\beta]_{h^*F} & = f\left(L_{\alpha\beta}^\gamma \circ h\right) T_\gamma + \left(\rho_\alpha^i \circ h\right) \dfrac{\partial f}{\partial x^i} T_\beta, \\ [fT_\alpha, T_\beta]_{h^*F} & = -[T_\beta, fT_\alpha]_{h^*F}, \end{array}
$$

for any $f \in \mathcal{F}(M)$, *it results that*

$$
\left((h^*F, h^*\nu, M), [,]_{h^*F}, \left(\overset{h^*F}{\rho}, Id_M\right)\right)
$$

is a Lie algebroid which is called the pull-back Lie algebroid of the generalized Lie algebroid $\left((F, \nu, M), [,]_{F,h}, (\rho, \eta)\right)$.

3 The Lie algebroid generalized tangent bundle

We consider the following diagram:

$$
(3.1) \qquad \begin{array}{ccc} E & & \left(F, [,]_{F,h}, (\rho, \eta)\right) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \nu} \\ M & \xrightarrow{\quad h \quad} & N \end{array}
$$

where (E, π, M) is a fiber bundle and $\left((F, \nu, M), [,]_{F,h}, (\rho, \eta)\right)$ is a generalized Lie algebroid.

We take $\left(x^i, y^a\right)$ as canonical local coordinates on (E, π, M), where $i \in \overline{1, m}$ and $a \in \overline{1, r}$. Let

$$
\left(x^i, y^a\right) \longrightarrow \left(x^{\acute{i}}\left(x^i\right), y^{\acute{a}}\left(x^i, y^a\right)\right)
$$

be a change of coordinates on (E, π, M). Then the coordinates y^a change to $y^{\acute{a}}$ according to the rule:

$$
(3.2) \qquad y^{\acute{a}} = \frac{\partial y^{\acute{a}}}{\partial y^a} y^a.
$$

In particular, if (E, π, M) is vector bundle, then the coordinates y^a change to $y^{\acute{a}}$ according to the rule:

$$
(3.2') \qquad y^{\acute{a}} = M_a^{\acute{a}} y^a.
$$

Let

$$\left(\pi^*\left(h^*F\right),\pi^*\left(h^*\nu\right),E\right),[,]_{\pi^*\left(h^*F\right)},\left(\overset{\pi^*\left(h^*F\right)}{\rho},Id_E\right)$$

be the pull-back Lie algebroid of the Lie algebroid

$$\left(h^*F,h^*\nu,M\right),[,]_{h^*F},\left(\overset{h^*F}{\rho},Id_M\right).$$

If $z=z^{\alpha}t_{\alpha}\in\Gamma\left(F,\nu,M\right)$, then, using the vector fibred $(m+r)+p$-atlas $\widetilde{\mathcal{AF}}_{\pi^*\left(h^*F\right)}$, we obtain the section

$$\tilde{Z}=\left(z^{\alpha}\circ h\circ\pi\right)\tilde{T}_{\alpha}\in\Gamma\left(\pi^*\left(h^*F\right),\pi^*\left(h^*\nu\right),E\right)$$

so that $\tilde{Z}\left(u_x\right)=z\left(h\left(x\right)\right)$, for any $u_x\in\pi^{-1}\left(U\cap h^{-1}V\right)$.

For any sections

$$\tilde{Z}^{\alpha}\tilde{T}_{\alpha}\in\Gamma\left(\pi^*\left(h^*F\right),\pi^*\left(h^*F\right),E\right)$$

and

$$Y^a\frac{\partial}{\partial y^a}\in\Gamma\left(VTE,\tau_E,E\right)$$

we obtain the section

$$\begin{aligned}\tilde{Z}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y^a\frac{\partial}{\partial\tilde{y}^a}&=:\tilde{Z}^{\alpha}\left(\tilde{T}_{\alpha}\oplus\left(\rho_{\alpha}^i\circ h\circ\pi\right)\frac{\partial}{\partial x^i}\right)+Y^a\left(0_{\pi^*\left(h^*F\right)}\oplus\frac{\partial}{\partial y^a}\right)\\&=\tilde{Z}^{\alpha}\tilde{T}_{\alpha}\oplus\left(\tilde{Z}^{\alpha}\left(\rho_{\alpha}^i\circ h\circ\pi\right)\frac{\partial}{\partial x^i}+Y^a\frac{\partial}{\partial y^a}\right)\in\Gamma\left(\pi^*\left(h^*F\right)\oplus TE,\overset{\oplus}{\pi},E\right).\end{aligned}$$

Since we have

$$\begin{gathered}\tilde{Z}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y^a\frac{\partial}{\partial\tilde{y}^a}=0\\\Updownarrow\\\tilde{Z}^{\alpha}\tilde{T}_{\alpha}=0\ \wedge\tilde{Z}^{\alpha}\left(\rho_{\alpha}^i\circ h\circ\pi\right)\frac{\partial}{\partial x^i}+Y^a\frac{\partial}{\partial y^a}=0,\end{gathered}$$

it implies $\tilde{Z}^{\alpha}=0,\ \alpha\in\overline{1,p}$ and $Y^a=0,\ a\in\overline{1,r}$.

Therefore, the sections $\frac{\partial}{\partial\tilde{z}^1},...,\frac{\partial}{\partial\tilde{z}^p},\frac{\partial}{\partial\tilde{y}^1},...,\frac{\partial}{\partial\tilde{y}^r}$ are linearly independent.

We consider the vector subbundle $\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$ of the vector bundle $\left(\pi^*\left(h^*F\right)\oplus TE,\overset{\oplus}{\pi},E\right)$, for which the $\mathcal{F}\left(E\right)$-module of sections is the $\mathcal{F}\left(E\right)$-submodule of $\left(\Gamma\left(\pi^*\left(h^*F\right)\oplus TE,\overset{\oplus}{\pi},E\right),+,\cdot\right)$, generated by the set of sections $\left(\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{y}^a}\right)$.

The base sections

$$\left(\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{y}^a}\right)\overset{put}{=}\left(\tilde{\partial}_{\alpha},\dot{\tilde{\partial}}_a\right)\tag{3.4}$$

will be called the *natural* (ρ,η)*-base.*

The matrix of coordinate transformation on $\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$ at a change of fibred charts is

$$\left\Vert\begin{array}{cc}\Lambda_{\alpha}^{\acute{\alpha}}\circ h\circ\pi&0\\\left(\rho_{\alpha}^i\circ h\circ\pi\right)\frac{\partial y^{\acute{a}}}{\partial x^i}&\frac{\partial y^{\acute{a}}}{\partial y^a}\end{array}\right\Vert.\tag{3.5}$$

In particular, if (E,π,M) is a vector bundle, then the matrix of coordinate transformation on $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ at a change of fibred charts is

$$\left\Vert \begin{array}{cc} \Lambda_{\alpha}^{\acute{\alpha}}\circ h\circ\pi & 0\\ \left(\rho_a^i\circ h\circ\pi\right)\dfrac{\partial M_b^{\acute{a}}\circ\pi}{\partial x_i}y^b & M_a^{\acute{a}}\circ\pi \end{array}\right\Vert. \tag{3.6}$$

Easily we obtain

Theorem 3.1 *Let $(\tilde{\rho},Id_E)$ be the* $\mathbf{B^v}$*-morphism of $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ source and (TE,τ_E,E) target, where*

$$\begin{array}{c} (\rho,\eta)TE\xrightarrow{\tilde{\rho}}TE\\ \left(\tilde{Z}^\alpha\dfrac{\partial}{\partial\tilde{z}^\alpha}+Y^a\dfrac{\partial}{\partial\tilde{y}^a}\right)(u_x)\longmapsto\left(\tilde{Z}^\alpha\left(\rho_\alpha^i\circ h\circ\pi\right)\dfrac{\partial}{\partial x^i}+Y^a\dfrac{\partial}{\partial y^a}\right)(u_x) \end{array} \tag{3.7}$$

Using the operation

$$\Gamma\left((\rho,\eta)TE,(\rho,\eta)\tau_E,E\right)^2\xrightarrow{[,]_{(\rho,\eta)TE}}\Gamma\left((\rho,\eta)TE,(\rho,\eta)\tau_E,E\right)$$

defined by

$$\begin{array}{c} \left[\left(\tilde{Z}_1^\alpha\dfrac{\partial}{\partial\tilde{z}^\alpha}+Y_1^a\dfrac{\partial}{\partial\tilde{y}^a}\right),\left(\tilde{Z}_2^\beta\dfrac{\partial}{\partial\tilde{z}^\beta}+Y_2^b\dfrac{\partial}{\partial\tilde{y}^b}\right)\right]_{(\rho,\eta)TE}\\ =\left[\tilde{Z}_1^\alpha\tilde{T}_a,\tilde{Z}_2^\beta\tilde{T}_\beta\right]_{\pi^*(h^*F)}\oplus\left[\left(\rho_\alpha^i\circ h\circ\pi\right)\tilde{Z}_1^\alpha\dfrac{\partial}{\partial x^i}+Y_1^a\dfrac{\partial}{\partial y^a},\right.\\ \left.\left(\rho_\beta^i\circ h\circ\pi\right)\tilde{Z}_2^\beta\dfrac{\partial}{\partial x^i}+Y_2^b\dfrac{\partial}{\partial y^b}\right]_{TE}, \end{array} \tag{3.8}$$

for any $\left(\tilde{Z}_1^\alpha\dfrac{\partial}{\partial\tilde{z}^\alpha}+Y_1^a\dfrac{\partial}{\partial\tilde{y}^a}\right)$ and $\left(\tilde{Z}_2^\beta\dfrac{\partial}{\partial\tilde{z}^\beta}+Y_2^b\dfrac{\partial}{\partial\tilde{y}^b}\right)$, we obtain that the couple

$$\left([,]_{(\rho,\eta)TE},(\tilde{\rho},Id_E)\right)$$

is a Lie algebroid structure for the vector bundle $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$.

Remark 3.2 In particular, if $h=Id_M$ and $[,]_{TM}$ is the usual Lie bracket, it results that the Lie algebroid

$$\left(\left((Id_{TM},Id_M)TE,(Id_{TM},Id_M)\tau_E,E\right),[,]_{(Id_{TM},Id_M)TE},\left(\widetilde{Id_{TM}},Id_E\right)\right)$$

is isomorphic with the usual Lie algebroid

$$\left((TE,\tau_E,E),[,]_{TE},(Id_{TE},Id_E)\right).$$

This is a reason for which the Lie algebroid

$$\left(((\rho,\eta)TE,(\rho,\eta)\tau_E,E),[,]_{(\rho,\eta)TE},(\tilde{\rho},Id_E)\right),$$

will be called the *Lie algebroid generalized tangent bundle.*

The vector bundle $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ will be called the *generalized tangent bundle.*

3.1 The generalized tangent bundle of dual vector bundle

Let (E, π, M) be a vector bundle. We build the generalized tangent bundle of dual vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ using the diagram:

$$\begin{array}{lcccc} & \overset{*}{E} & & \left(F, [,]_{F,h}, (\rho, \eta)\right) & \\ (3.1.1) & \overset{*}{\pi} \downarrow & & \downarrow \nu & , \\ & M & \xrightarrow{h} & N & \end{array}$$

where $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$ is a generalized Lie algebroid.

We take $\left(x^i, p_a\right)$ as canonical local coordinates on $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, where $i \in \overline{1, m}$ and $a \in \overline{1, r}$.

Consider

$$\left(x^i, p_a\right) \longrightarrow \left(x^{\acute{i}}\left(x^i\right), p_{\acute{a}}\left(x^i, p_a\right)\right)$$

a change of coordinates on $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$. Then the coordinates p_a change to $p_{\acute{a}}$ according to the rule:

$$p_{\acute{a}} = M_{\acute{a}}^{a} p_a. \tag{3.1.2}$$

Let

$$\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial p_a}\right)$$

be the natural base for the sections Lie algebra $\left(\Gamma\left(T\overset{*}{E}, \tau_{\overset{*}{E}}, \overset{*}{E}\right), +, \cdot, [,]_{T\overset{*}{E}}\right)$.

For any sections

$$\tilde{Z}^\alpha \tilde{T}_\alpha \in \Gamma\left(\overset{*}{\pi}{}^{*}\left(h^* F\right), \overset{*}{\pi}{}^{*}\left(h^* \nu\right), \overset{*}{E}\right)$$

and

$$Y_a \tfrac{\partial}{\partial p_a} \in \Gamma\left(VT\overset{*}{E}, \tau_{\overset{*}{E}}, \overset{*}{E}\right),$$

we construct the section

$$\begin{aligned} \tilde{Z}^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y_a \frac{\partial}{\partial \tilde{p}_a} &=: \tilde{Z}^\alpha\left(T_\alpha \oplus \left(\rho_\alpha^i \circ h \circ \overset{*}{\pi}\right) \frac{\partial}{\partial x^i}\right) + Y_a\left(0_{\overset{*}{\pi}{}^{*}\left(h^* F\right)} \oplus \frac{\partial}{\partial p_a}\right) \\ &= \tilde{Z}^\alpha \tilde{T}_\alpha \oplus \left(\tilde{Z}^\alpha\left(\rho_\alpha^i \circ h \circ \overset{*}{\pi}\right) \frac{\partial}{\partial x^i} + Y_a \frac{\partial}{\partial p_a}\right) \in \Gamma\left(\overset{*}{\pi}{}^{*}\left(h^* F\right) \oplus T\overset{*}{E}, \overset{\oplus}{\overset{*}{\pi}}, \overset{*}{E}\right). \end{aligned}$$

Since we have

$$\begin{gathered} \tilde{Z}^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y_a \frac{\partial}{\partial \tilde{p}_a} = 0_{\overset{*}{\pi}{}^{*}\left(h^* F\right) \oplus T\overset{*}{E}} \\ \Updownarrow \\ \tilde{Z}^\alpha \tilde{T}_\alpha = 0_{\overset{*}{\pi}{}^{*}\left(h^* F\right)} \wedge \tilde{Z}^\alpha\left(\rho_\alpha^i \circ h \circ \overset{*}{\pi}\right) \frac{\partial}{\partial x^i} + Y_a \frac{\partial}{\partial p_a} = 0_{T\overset{*}{E}}, \end{gathered}$$

it implies $\tilde{Z}^\alpha = 0,\ \alpha \in \overline{1, p}$ and $Y_a = 0,\ a \in \overline{1, r}$.

Therefore, the sections

$$\frac{\partial}{\partial \tilde{z}^{1}}, ..., \frac{\partial}{\partial \tilde{z}^{p}}, \frac{\partial}{\partial \tilde{p}_{1}}, ..., \frac{\partial}{\partial \tilde{p}_{r}}$$

are linearly independent.

We consider the vector subbundle

$$\left((\rho, \eta)\, T\overset{*}{E}, (\rho, \eta)\, \tau_{\overset{*}{E}}, \overset{*}{E} \right)$$

of vector bundle

$$\left(\overset{*}{\pi}^{*}\left(h^{*}F\right) \oplus T\overset{*}{E}, \overset{\oplus}{\overset{*}{\pi}}, \overset{*}{E} \right),$$

for which the $\mathcal{F}\left(\overset{*}{E}\right)$-module of sections is the $\mathcal{F}\left(\overset{*}{E}\right)$-submodule of

$$\left(\Gamma \left(\overset{*}{\pi}^{*}\left(h^{*}F\right) \oplus T\overset{*}{E}, \overset{\oplus}{\overset{*}{\pi}}, \overset{*}{E} \right), +, \cdot \right),$$

generated by the family of sections $\left(\frac{\partial}{\partial \tilde{z}^{\alpha}}, \frac{\partial}{\partial \tilde{p}_{a}} \right)$.

The base sections

$$\left(\tfrac{\partial}{\partial \tilde{z}^{\alpha}}, \tfrac{\partial}{\partial \tilde{p}_{a}} \right) \overset{put}{=} \left(\tilde{\partial}_{\alpha}, \dot{\tilde{\partial}}^{a} \right) \tag{3.1.3}$$

will be called the *natural* (ρ, η)*-base.*

The matrix of coordinate transformation on $\left((\rho, \eta)\, T\overset{*}{E}, (\rho, \eta)\, \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ at a change of fibred charts is

$$\left\Vert \begin{array}{cc} \Lambda_{\alpha}^{\acute{\alpha}} \circ h \circ \overset{*}{\pi} & 0 \\ \left(\rho_{\alpha}^{i} \circ h \circ \overset{*}{\pi}\right) \dfrac{\partial M_{\acute{a}}^{b} \circ \overset{*}{\pi}}{\partial x_{i}} p_{b} & M_{\acute{a}}^{a} \circ \overset{*}{\pi} \end{array} \right\Vert . \tag{3.1.4}$$

We consider the operation $[,]_{(\rho,\eta)T\overset{*}{E}}$ defined by

$$\begin{array}{l} \left[\left(\tilde{Z}_{1}^{\alpha}\dfrac{\partial}{\partial \tilde{z}^{\alpha}} + Y_{1a}\dfrac{\partial}{\partial \tilde{p}_{a}}\right), \left(\tilde{Z}_{2}^{\beta}\dfrac{\partial}{\partial \tilde{z}^{\beta}} + Y_{2b}\dfrac{\partial}{\partial \tilde{p}_{b}}\right) \right]_{(\rho,\eta)TE^{*}} = \\ = \left[\tilde{Z}_{1}^{\alpha}T_{a}, \tilde{Z}_{2}^{\beta}T_{\beta}\right]_{\overset{*}{\pi}^{*}(h^{*}F)} \oplus \left[\left(\rho_{\alpha}^{i} \circ h \circ \pi^{*}\right) \tilde{Z}_{1}^{\alpha}\dfrac{\partial}{\partial x^{i}} + Y_{1a}\dfrac{\partial}{\partial p_{a}}, \right. \\ \qquad \left. \left(\rho_{\beta}^{i} \circ h \circ \pi^{*}\right) \tilde{Z}_{2}^{\beta}\dfrac{\partial}{\partial x^{i}} + Y_{2b}\dfrac{\partial}{\partial p_{b}} \right]_{T\overset{*}{E}}, \end{array} \tag{3.1.5}$$

for any sections $\left(\tilde{Z}_{1}^{\alpha}\frac{\partial}{\partial \tilde{z}^{\alpha}} + Y_{1a}\frac{\partial}{\partial \tilde{p}_{a}}\right)$ and $\left(\tilde{Z}_{2}^{\beta}\frac{\partial}{\partial \tilde{z}^{\beta}} + Y_{2b}\frac{\partial}{\partial \tilde{p}_{b}}\right)$.

Let $\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)$ be the $\mathbf{B^v}$-morphism of $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ source and $\left(T\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right)$ target, where

$$\begin{array}{c}(\rho,\eta)\,T\overset{*}{E}\xrightarrow{\tilde{\rho}^{*}}T\overset{*}{E}\\ \left(\tilde{Z}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y_a\frac{\partial}{\partial\tilde{p}_a}\right)(\overset{*}{u}_x)\longmapsto\left(\tilde{Z}^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\frac{\partial}{\partial x^{i}}+Y_a\frac{\partial}{\partial p_a}\right)(\overset{*}{u}_x)\end{array}\tag{3.1.6}$$

The Lie algebroid generalized tangent bundle of the dual vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ will be denoted

$$\left(\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),[,]_{(\rho,\eta)T\overset{*}{E}},\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\right).$$

4 (Linear) (ρ,η)-connections

We consider the diagram:

$$\begin{array}{ccc} E & & \left(F,[,]_{F,h},(\rho,\eta)\right) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow{\quad h\quad} & N \end{array}$$

where $(E,\pi,M)\in|\mathbf{B}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Let

$$\left(((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E),[,]_{(\rho,\eta)TE},(\tilde{\rho},Id_E)\right)$$

be the Lie algebroid generalized tangent bundle of the fiber bundle (E,π,M).

We consider the $\mathbf{B^v}$-morphism $((\rho,\eta)\,\pi!,Id_E)$ given by the commutative diagram

$$\begin{array}{ccc} (\rho,\eta)\,TE & \xrightarrow{(\rho,\eta)\pi!} & \pi^{*}(h^{*}F) \\ \downarrow{\scriptstyle(\rho,\eta)\tau_E} & & \downarrow{\scriptstyle pr_1} \\ E & \xrightarrow{\quad id_E\quad} & E \end{array}\tag{4.1}$$

This is defined as:

$$(\rho,\eta)\,\pi!\left(\left(\tilde{Z}^{\alpha}\tfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y^{a}\tfrac{\partial}{\partial\tilde{y}^{a}}\right)(u_x)\right)=\left(\tilde{Z}^{\alpha}\tilde{T}_{\alpha}\right)(u_x),\tag{4.2}$$

for any $\left(\tilde{Z}^{\alpha}\dfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y^{a}\dfrac{\partial}{\partial\tilde{y}^{a}}\right)\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$.

Using the $\mathbf{B^v}$-morphism $((\rho,\eta)\,\pi!,Id_E)$, and the the $\mathbf{B^v}$-morphism (2.7) we obtain the *tangent* (ρ,η)*-application* $((\rho,\eta)\,T\pi,h\circ\pi)$ of $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ source and (F,ν,N) target.

Definition 4.1 The kernel of the tangent (ρ,η)-application is writen

$$(V\,(\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$$

and it is called *the vertical subbundle.*

We remark that the set $\left\{\frac{\partial}{\partial\tilde{y}^a},\ a\in\overline{1,r}\right\}$ is a base of the $\mathcal{F}(E)$-module

$$\left(\Gamma\left(V\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right),+,\cdot\right).$$

Proposition 4.1 *The short sequence of vector bundles*

$$\begin{tikzcd} 0 \arrow[r, hook, "i"] \arrow[d] & V(\rho,\eta)TE \arrow[r, hook, "i"] \arrow[d] & (\rho,\eta)TE \arrow[r, "(\rho,\eta)\pi!"] \arrow[d] & \pi^*\left(h^*F\right) \arrow[r] \arrow[d] & 0 \arrow[d] \\ E \arrow[r, "Id_E"] & E \arrow[r, "Id_E"] & E \arrow[r, "Id_E"] & E \arrow[r, "Id_E"] & E \end{tikzcd} \tag{4.3}$$

is exact.

Definition 4.2 *A* **Man**-morphism $\left(\rho,\eta\right)\Gamma$ of $\left(\rho,\eta\right)TE$ source and $V\left(\rho,\eta\right)TE$ target defined by

$$\left(\rho,\eta\right)\Gamma\left(\tilde{Z}^\alpha\frac{\partial}{\partial\tilde{z}^\alpha}+Y^a\frac{\partial}{\partial\tilde{y}^a}\right)\left(u_x\right)=\left(Y^a+\left(\rho,\eta\right)\Gamma_\alpha^a\tilde{Z}^\alpha\right)\frac{\partial}{\partial\tilde{y}^a}\left(u_x\right), \tag{4.4}$$

so that the $\mathbf{B^v}$-morphism $\left(\left(\rho,\eta\right)\Gamma,Id_E\right)$ is a split to the left in the previous exact sequence, will be called (ρ,η)-*connection for the fiber bundle* (E,π,M).

The (ρ,Id_M)-connection will be called ρ-*connection* and will be denoted $\rho\Gamma$ and the (Id_{TM},Id_M)-connection will be called *connection* and will be denoted Γ.

Definition 4.3 If $\left(\rho,\eta\right)\Gamma$ is a (ρ,η)-connection for the fiber bundle (E,π,M), then the kernel of the $\mathbf{B^v}$-morphism $\left(\left(\rho,\eta\right)\Gamma,Id_E\right)$ is written $\left(H\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$ and will be called the *horizontal vector subbundle.*

Definition 4.4 If $(E,\pi,M)\in|\mathbf{B}|$, then the **B**-morphism (Π,π) defined by the commutative diagram

$$\begin{tikzcd} V\left(\rho,\eta\right)TE \arrow[r, "\Pi"] \arrow[d, "(\rho,\eta)\tau_E"'] & E \arrow[d, "\pi"] \\ E \arrow[r, "\pi"] & M \end{tikzcd} \tag{4.5}$$

so that the components of the image of the vector $Y^a\frac{\partial}{\partial\tilde{y}^a}\left(u_x\right)$ are the real numbers $Y^1\left(u_x\right),...,Y^r\left(u_x\right)$ will be called the *canonical projection* **B**-*morphism.*

In particular, if $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left\{s_a,a\in\overline{1,r}\right\}$ is a base of the $\mathcal{F}(M)$-module of sections $\left(\Gamma\left(E,\pi,M\right),+,\cdot\right)$, then Π is defined by

$$\Pi\left(Y^a\frac{\partial}{\partial\tilde{y}^a}\left(u_x\right)\right)=Y^a\left(u_x\right)s_a\left(x\right). \tag{4.6}$$

Theorem 4.1 *If* $\left(\rho,\eta\right)\Gamma$ *is a* (ρ,η)*-connection for the fiber bundle* $\left(E,\pi,M\right)$, *then its components satisfy the law of transformation*

$$\left(\rho,\eta\right)\Gamma_{\gamma'}^{a'}=\frac{\partial y^{a'}}{\partial y^a}\left[\rho_\gamma^i\circ\left(h\circ\pi\right)\frac{\partial y^a}{\partial x^i}+\left(\rho,\eta\right)\Gamma_\gamma^a\right]\Lambda_{\gamma'}^\gamma\circ\left(h\circ\pi\right). \tag{4.7}$$

If $\left(\rho,\eta\right)\Gamma$ *is a* (ρ,η)*-connection for the vector bundle* $\left(E,\pi,M\right)$, *then its components satisfy the law of transformation*

$$\left(\rho,\eta\right)\Gamma_{\gamma'}^{a'}=M_a^{a'}\circ\pi\left[\rho_\gamma^i\circ\left(h\circ\pi\right)\frac{\partial M_{b'}^a\circ\pi}{\partial x^i}y^{b'}+\left(\rho,\eta\right)\Gamma_\gamma^a\right]\Lambda_{\gamma'}^\gamma\circ\left(h\circ\pi\right). \tag{4.7'}$$

If $\rho\Gamma$ is a ρ-connection for the vector bundle (E,π,M) and $h=Id_M$, then relations $(4.7')$ *become*

$$(4.7'') \qquad \rho\Gamma_{\acute{\gamma}}^{\acute{a}} = M_a^{\acute{a}}\circ\pi\left[\rho_{\acute{\gamma}}^{i}\circ\pi\tfrac{\partial M_{\acute{b}}^{a}\circ\pi}{\partial x^i}y^{\acute{b}}+\rho\Gamma_{\gamma}^{a}\right]\Lambda_{\acute{\gamma}}^{\gamma}\circ\pi.$$

In particular, if $\rho=Id_{TM}$, *then the relations* $(4.7'')$ *become*

$$(4.7''') \qquad \Gamma_{\acute{k}}^{\acute{i}} = \tfrac{\partial x^{\acute{i}}}{\partial x^i}\circ\pi\left[\tfrac{\partial}{\partial x^k}\left(\tfrac{\partial x^i}{\partial x^{\acute{j}}}\circ\pi\right)y^{\acute{j}}+\Gamma_k^i\right]\tfrac{\partial x^k}{\partial x^{\acute{k}}}\circ\pi.$$

Proof. Let (Π,π) be the canonical projection **B**-morphism.

Obviously, the components of

$$\Pi\circ(\rho,\eta)\Gamma\left(\tilde{Z}^{\acute{\alpha}}\tfrac{\partial}{\partial\tilde{z}^{\acute{\alpha}}}+Y^{\acute{a}}\tfrac{\partial}{\partial\tilde{y}^{\acute{a}}}\right)(u_x)$$

are the real numbers

$$\left(Y^{\acute{a}}+(\rho,\eta)\Gamma_{\acute{\gamma}}^{\acute{a}}\tilde{Z}^{\acute{\gamma}}\right)(u_x).$$

Since

$$\begin{array}{c}\left(\tilde{Z}^{\acute{\alpha}}\dfrac{\partial}{\partial\tilde{z}^{\acute{\alpha}}}+Y^{\acute{a}}\dfrac{\partial}{\partial\tilde{y}^{\acute{a}}}\right)(u_x)=\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi\dfrac{\partial}{\partial\tilde{z}^{\alpha}}(u_x)\\ +\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{i}}\circ h\circ\pi\dfrac{\partial y^a}{\partial x^{\acute{i}}}+\dfrac{\partial y^a}{\partial y^{\acute{a}}}Y^{\acute{a}}\right)\dfrac{\partial}{\partial\tilde{y}^a}(u_x),\end{array}$$

it results that the components of

$$\Pi\circ(\rho,\eta)\Gamma\left(\tilde{Z}^{\acute{\alpha}}\tfrac{\partial}{\partial\tilde{z}^{\acute{\alpha}}}+Y^{\acute{a}}\tfrac{\partial}{\partial\tilde{y}^{\acute{a}}}\right)(u_x)$$

are the real numbers

$$\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{i}}\circ h\circ\pi\tfrac{\partial y^a}{\partial x^{\acute{i}}}+\tfrac{\partial y^a}{\partial y^{\acute{a}}}Y^{\acute{a}}+(\rho,\eta)\Gamma_{\alpha}^{a}\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi\right)(u_x)\tfrac{\partial y^{\acute{a}}}{\partial y^a},$$

where

$$\left\|\tfrac{\partial y^a}{\partial y^{\acute{a}}}\right\|=\left\|\tfrac{\partial y^{\acute{a}}}{\partial y^a}\right\|^{-1}.$$

Therefore, we have:

$$\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{i}}\circ h\circ\pi\tfrac{\partial y^a}{\partial x^{\acute{i}}}+\tfrac{\partial y^a}{\partial y^{\acute{a}}}Y^{\acute{a}}+(\rho,\eta)\Gamma_{\alpha}^{a}\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi\right)\tfrac{\partial y^{\acute{a}}}{\partial y^a}=Y^{\acute{a}}+(\rho,\eta)\Gamma_{\acute{\alpha}}^{\acute{a}}\tilde{Z}^{\acute{\alpha}}.$$

After some calculations we obtain:

$$(\rho,\eta)\Gamma_{\acute{\alpha}}^{\acute{a}}=\tfrac{\partial y^{\acute{a}}}{\partial y^a}\left(\rho_{\alpha}^{i}\circ(h\circ\pi)\tfrac{\partial y^a}{\partial x^i}+(\rho,\eta)\Gamma_{\alpha}^{a}\right)\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi. \qquad \textit{q.e.d.}$$

Remark 4.1 If Γ is a Ehresmann connection for the vector bundle (E,π,M) on components Γ_k^a, then the differentiable real local functions $(\rho,\eta)\Gamma_{\gamma}^{a}=\left(\rho_{\gamma}^{k}\circ h\circ\pi\right)\Gamma_k^a$ are the components of a (ρ,η)-connection $(\rho,\eta)\Gamma$ for the vector bundle (E,π,M). This (ρ,η)-connection will be called the (ρ,η)*-connection associated to the Eresmann connection* Γ.

Definition 4.5 If $(\rho,\eta)\Gamma$ is a (ρ,η)-connection for the vector bundle (E,π,M) and $z=z^{\alpha}t_{\alpha}\in\Gamma(F,\nu,M)$, then the application

$$\begin{array}{ccc}\Gamma(E,\pi,M) & \xrightarrow{(\rho,\eta)D_z} & \Gamma(E,\pi,M)\\ u=u^a s_a & \longmapsto & (\rho,\eta)D_z u\end{array}$$

where

$$(4.8)\qquad (\rho,\eta)\,D_z u = z^\alpha \circ h\left(\rho_\alpha^i \circ h \frac{\partial u^a}{\partial x^i} + (\rho,\eta)\,\Gamma_\alpha^a \circ u\right) s_a$$

will be called the *covariant* (ρ,η)*-derivative associated to* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$ *with respect to the section* z.

Definition 4.6 Let $(\rho,\eta)\,\Gamma$ be a (ρ,η)-connection for the fiber bundle (E,π,M). If for each local vector $(m+r)$-chart (U,s_U) and for each local vector $(n+p)$-chart (V,t_V) so that $U\cap h^{-1}(V)\neq\phi$, it exists the differentiable real functions $(\rho,\eta)\,\Gamma_{b\gamma}^a$ defined on $U\cap h^{-1}(V)$ such that

$$(4.9)\qquad (\rho,\eta)\,\Gamma_\gamma^a \circ u = (\rho,\eta)\,\Gamma_{b\gamma}^a \cdot u^b, \forall u = u^b s_b \in \Gamma(E,\pi,M),$$

then we say that $(\rho,\eta)\,\Gamma$ *is linear.*

The differentiable real local functions $(\rho,\eta)\,\Gamma_{b\alpha}^a$ will be called the *Christoffel coefficients of linear* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$.

Proposition 4.2 *If* $(\rho,\eta)\,\Gamma$ *is a linear* (ρ,η)*-connection for the fiber bundle* (E,π,M), *then its components satisfy the law of transformation*

$$(4.10)\qquad (\rho,\eta)\Gamma_{b\acute{}\gamma\acute{}}^{a\acute{}}=\frac{\partial y^{a\acute{}}}{\partial y^a}\left[\rho_\gamma^k\circ h\frac{\partial}{\partial x^k}\left(\frac{\partial y^a}{\partial y^{b\acute{}}}\right)+(\rho,\eta)\,\Gamma_{b\gamma}^a\frac{\partial y^b}{\partial y^{b\acute{}}}\right]\Lambda_{\gamma\acute{}}^\gamma\circ h.$$

If $(\rho,\eta)\,\Gamma$ *is a linear* (ρ,η)*-connection for the vector bundle* (E,π,M), *then its components satisfy the law of transformation*

$$(4.10')\qquad (\rho,\eta)\Gamma_{b\acute{}\gamma\acute{}}^{a\acute{}}=M_a^{a\acute{}}\left[\rho_\gamma^k\circ h\frac{\partial M_{b\acute{}}^a}{\partial x^k}+(\rho,\eta)\Gamma_{b\gamma}^a M_{b\acute{}}^b\right]\Lambda_{\gamma\acute{}}^\gamma\circ h.$$

If $\rho\Gamma$ *is a* ρ*-connection for the vector bundle* (E,π,M) *and* $h=Id_M$, *then the relations* $(4.10')$ *become*

$$(4.10'')\qquad \rho\Gamma_{b\acute{}\gamma\acute{}}^{a\acute{}} = M_a^{a\acute{}}\left[\rho_\gamma^k\frac{\partial M_{b\acute{}}^a}{\partial x^k}+\rho\Gamma_{b\gamma}^a M_{b\acute{}}^b\right]\Lambda_{\gamma\acute{}}^\gamma.$$

In particular, if $\rho = Id_{TM}$, *then the relations* $(4.10'')$ *become*

$$(4.10''')\qquad \Gamma_{j\acute{}k\acute{}}^{i\acute{}} = \frac{\partial x^{i\acute{}}}{\partial x^i}\left[\frac{\partial}{\partial x^k}\left(\frac{\partial x^i}{\partial x^{j\acute{}}}\right)+\Gamma_{jk}^i\frac{\partial x^j}{\partial x^{j\acute{}}}\right]\frac{\partial x^k}{\partial x^{k\acute{}}}.$$

Remark 4.2 If $(\rho,\eta)\,\Gamma$ is a linear (ρ,η)-connection for the vector bundle (E,π,M), then, for any

$$z = z^\alpha t_\alpha \in \Gamma(F,\nu,M),$$

we obtain the *covariant* (ρ,η)*-derivative associated to the linear* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$ *with respect to the section* z

$$\begin{array}{ccc}\Gamma(E,\pi,M) & \xrightarrow{(\rho,\eta)D_z} & \Gamma(E,\pi,M)\\ u=u^a s_a & \longmapsto & (\rho,\eta)D_z u\end{array}$$

defined by

$$(4.11)\qquad (\rho,\eta)D_z u=z^\alpha\circ h\left(\rho_\alpha^i\circ h\frac{\partial u^a}{\partial x^i}+(\rho,\eta)\Gamma_{b\alpha}^a\cdot u^b\right)s_a.$$

4.1 (Linear) (ρ,η)-connections for dual vector bundle

Let (E,π,M) be a vector bundle.

We consider the following diagram:

$$\begin{array}{lccc} & \overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right) \\ (4.1.1) & \overset{*}{\pi}\downarrow & & \downarrow\nu \\ & M & \xrightarrow{h} & N \end{array},$$

where $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Let

$$\left(\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right),[,]_{(\rho,\eta)T\overset{*}{E}},\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\right)$$

be the Lie algebroid generalized tangent bundle of the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

We consider the $\mathbf{B^v}$-morphism $\left((\rho,\eta)\overset{*}{\pi}!,Id_{\overset{*}{E}}\right)$ given by the commutative diagram

$$\begin{array}{ccc} (\rho,\eta)T\overset{*}{E} & \xrightarrow{(\rho,\eta)\overset{*}{\pi}!} & \overset{*}{\pi}{}^{*}(h^{*}F) \\ {\scriptstyle (\rho,\eta)\tau_{\overset{*}{E}}}\downarrow & & \downarrow{\scriptstyle pr_1} \\ \overset{*}{E} & \xrightarrow{id_{\overset{*}{E}}} & \overset{*}{E} \end{array} \tag{4.1.2}$$

Using the components, this is defined as:

$$(\rho,\eta)\overset{*}{\pi}!\left(\tilde{Z}^{\alpha}\tfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y_{a}\tfrac{\partial}{\partial\tilde{p}_{a}}\right)\left(\overset{*}{u}_{x}\right)=\left(\tilde{Z}^{\alpha}\tilde{T}_{\alpha}\right)\left(\overset{*}{u}_{x}\right), \tag{4.1.3}$$

for any $\tilde{Z}^{\alpha}\dfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y_{a}\dfrac{\partial}{\partial\tilde{p}_{a}}\in\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Using the $\mathbf{B^v}$-morphism $\left((\rho,\eta)\overset{*}{\pi}!,Id_{\overset{*}{E}}\right)$ and the $\mathbf{B^v}$-morphism (2.7) we obtain the *tangent* (ρ,η)*-application* $\left((\rho,\eta)T\overset{*}{\pi},h\circ\overset{*}{\pi}\right)$ of $\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$ source and (F,ν,N) target.

Definition 4.1.1 The kernel of the tangent (ρ,η)-application

$$\left((\rho,\eta)T\overset{*}{\pi},h\circ\overset{*}{\pi}\right)$$

is written as

$$\left(V(\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

and will be called the *vertical subbundle.*

The set $\left\{\dfrac{\partial}{\partial\tilde{p}_{a}},\ a\in\overline{1,r}\right\}$ is a base for the $\mathcal{F}\left(\overset{*}{E}\right)$-module

$$\left(\Gamma\left(V(\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right).$$

Proposition 4.1.1 *The short sequence of vector bundles*

$$(4.1.4)\qquad \begin{tikzcd} 0 \arrow[r, hook, "i"] \arrow[d] & V(\rho,\eta)T\overset{*}{E} \arrow[r, hook, "i"] \arrow[d] & (\rho,\eta)T\overset{*}{E} \arrow[r, "(\rho,\eta)\overset{*}{\pi}!"] \arrow[d] & \overset{*}{\pi}{}^{*}(h^*F) \arrow[r] \arrow[d] & 0 \arrow[d] \\ \overset{*}{E} \arrow[r, "Id_{\overset{*}{E}}"] & \overset{*}{E} \arrow[r, "Id_{\overset{*}{E}}"] & \overset{*}{E} \arrow[r, "Id_{\overset{*}{E}}"] & \overset{*}{E} \arrow[r, "Id_{\overset{*}{E}}"] & \overset{*}{E} \end{tikzcd}$$

is exact.

Definition 4.1.2 *A* **Man**-morphism $(\rho,\eta)\overset{*}{\Gamma}$ of $(\rho,\eta)T\overset{*}{E}$ source and $V(\rho,\eta)T\overset{*}{E}$ target defined by

$$(4.1.5)\qquad (\rho,\eta)\,\Gamma\left(\tilde{Z}^{\alpha}\tfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y_b\tfrac{\partial}{\partial\tilde{p}_b}\right)\left(\overset{*}{u}_x\right)=\left(Y_b-(\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}\tilde{Z}^{\alpha}\right)\tfrac{\partial}{\partial\tilde{p}_b}\left(\overset{*}{u}_x\right),$$

such that the $\mathbf{B^v}$-morphism $\left((\rho,\eta)\,\overset{*}{\Gamma},Id_{\overset{*}{E}}\right)$ is a split to the left in the previous exact sequence, will be called (ρ,η)*-connection for the dual vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

The differentiable real local functions $(\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}$ will be called the *components of* (ρ,η)*-connection* $(\rho,\eta)\,\overset{*}{\Gamma}$.

The (ρ,Id_M)-connection for the dual vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ will be called ρ*-connection for the dual vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and will be denoted $\rho\overset{*}{\Gamma}$.

The (Id_{TM},Id_M)-connection for the dual vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ will be called *connection for the dual vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and will be denoted $\overset{*}{\Gamma}$.

Let $\left\{s^a,\ a\in\overline{1,r}\right\}$ be the dual base of the base $\left\{s_a,a\in\overline{1,r}\right\}$.

The $\mathbf{B^v}$-morphism $\left(\overset{*}{\Pi},\overset{*}{\pi}\right)$ defined by the commutative diagram

$$(4.1.6)\qquad \begin{tikzcd} V(\rho,\eta)T\overset{*}{E} \arrow[r, "\overset{*}{\Pi}"] \arrow[d, "(\rho,\eta)\tau_{\overset{*}{E}}"'] & \overset{*}{E} \arrow[d, "\overset{*}{\pi}"] \\ \overset{*}{E} \arrow[r, "\overset{*}{\pi}"] & M \end{tikzcd},$$

where, $\overset{*}{\Pi}$ is defined by

$$(4.1.7)\qquad \overset{*}{\Pi}\left(Y_a\frac{\partial}{\partial\tilde{p}_a}\left(\overset{*}{u}_x\right)\right)=Y_a\left(\overset{*}{u}_x\right)s^a\left(\overset{*}{\pi}\left(\overset{*}{u}_x\right)\right),$$

is canonical projection $\mathbf{B^v}$-morphism.

Theorem 4.1.1 *If* $(\rho,\eta)\,\overset{*}{\Gamma}$ *is a* (ρ,η)*-connection for the vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, *then its components satisfy the law of transformation*

$$(4.1.8)\qquad (\rho,\eta)\,\overset{*}{\Gamma}_{b'\gamma'}=M_{b'}^{b}\circ\overset{*}{\pi}\left[-\rho_{\gamma}^{i}\circ h\circ\overset{*}{\pi}\cdot\tfrac{\partial M_{b}^{a'}\circ\overset{*}{\pi}}{\partial x^{i}}p_{a'}+(\rho,\eta)\,\overset{*}{\Gamma}_{b\gamma}\right]\Lambda_{\gamma'}^{\gamma}\circ\left(h\circ\overset{*}{\pi}\right).$$

In particular, if $h = Id_M$*, then the relations* (4.1.8) *become*

$$(4.1.8') \qquad (\rho,\eta)\overset{*}{\Gamma}_{b'\gamma'} = M_{b'}^{b}\circ\overset{*}{\pi}\left[-\rho_{\gamma}^{i}\circ\overset{*}{\pi}\cdot\frac{\partial M_{b}^{\alpha'}\circ\overset{*}{\pi}}{\partial x^{i}}p_{\alpha'} + (\rho,\eta)\overset{*}{\Gamma}_{b\gamma}\right]\Lambda_{\gamma'}^{\gamma}\circ\overset{*}{\pi}.$$

In particular, if $(\rho,\eta) = (Id_{TM}, Id_M)$*, then the relations* (4.1.8′) *become*

$$(4.1.8'') \qquad \overset{*}{\Gamma}_{j'k'} = \frac{\partial x^{j}}{\partial x^{j'}}\circ\overset{*}{\pi}\left[-\frac{\partial}{\partial x^{i}}\left(\frac{\partial x^{i'}}{\partial x^{j}}\circ\overset{*}{\pi}\right)p_{i'} + \overset{*}{\Gamma}_{jk}\right]\frac{\partial x^{k}}{\partial x^{k'}}\circ\overset{*}{\pi}.$$

Proof. Let $\left(\overset{*}{\Pi},\overset{*}{\pi}\right)$ be the canonical projection **B**-morphism.

Obviously, the components of

$$\Pi^{*}\circ(\rho,\eta)\overset{*}{\Gamma}\left(\tilde{Z}^{\alpha'}\frac{\partial}{\partial\tilde{z}^{\alpha'}} + Y_{b'}\frac{\partial}{\partial\tilde{p}_{b'}}\right)\left(\overset{*}{u}_{x}\right)$$

are the real numbers

$$\left(Y_{b'} - (\rho,\eta)\overset{*}{\Gamma}_{b'\gamma'}\tilde{Z}^{\gamma'}\right)\left(\overset{*}{u}_{x}\right).$$

Since

$$\begin{aligned}\left(\tilde{Z}^{\alpha'}\frac{\partial}{\partial\tilde{z}^{\alpha'}} + Y_{b'}\frac{\partial}{\partial\tilde{p}_{b'}}\right)\left(\overset{*}{u}_{x}\right) &= \tilde{Z}^{\alpha'}\Lambda_{\alpha'}^{\alpha}\circ h\circ\overset{*}{\pi}\frac{\partial}{\partial\tilde{z}^{\alpha}}\left(\overset{*}{u}_{x}\right)\\ &+\left(\tilde{Z}^{\alpha'}\rho_{\alpha'}^{i'}\circ h\circ\overset{*}{\pi}\frac{\partial M_{b}^{\alpha'}\circ\pi}{\partial x^{i'}}p_{\alpha'} + M_{b}^{b'}Y_{b'}\right)\frac{\partial}{\partial\tilde{p}_{b}}\left(\overset{*}{u}_{x}\right),\end{aligned}$$

it results that the components of

$$\Pi^{*}\circ(\rho,\eta)\overset{*}{\Gamma}\left(\tilde{Z}^{\alpha'}\frac{\partial}{\partial\tilde{z}^{\alpha'}} + Y_{b'}\frac{\partial}{\partial\tilde{p}_{b'}}\right)\left(\overset{*}{u}_{x}\right)$$

are the real numbers

$$\left(\tilde{Z}^{\alpha}\rho_{\alpha'}^{i'}\circ h\circ\overset{*}{\pi}\frac{\partial M_{b}^{\alpha'}\circ\overset{*}{\pi}}{\partial x^{i'}}p_{\alpha'} + M_{b}^{b'}\circ\overset{*}{\pi}Y_{b'} - (\rho,\eta)\overset{*}{\Gamma}_{b\alpha}\tilde{Z}^{\alpha'}\Lambda_{\alpha'}^{\alpha}\circ h\circ\overset{*}{\pi}\right)M_{b'}^{b}\circ\overset{*}{\pi}\left(\overset{*}{u}_{x}\right),$$

where $\left\|M_{b'}^{b}\right\| = \left\|M_{b}^{b'}\right\|^{-1}$.

Therefore, we have:

$$\begin{aligned}&\left(\tilde{Z}^{\alpha'}\rho_{\alpha'}^{i'}\circ h\circ\overset{*}{\pi}\frac{\partial M_{b}^{\alpha'}\circ\overset{*}{\pi}}{\partial x^{i'}}p_{\alpha'} + M_{b}^{b'}\circ\overset{*}{\pi}Y_{b'} - (\rho,\eta)\overset{*}{\Gamma}_{b\alpha}\tilde{Z}^{\alpha'}\Lambda_{\alpha'}^{\alpha}\circ h\circ\overset{*}{\pi}\right)M_{b'}^{b}\circ\overset{*}{\pi}\\ &\qquad= Y_{b'} - (\rho,\eta)\overset{*}{\Gamma}_{b'\alpha'}\tilde{Z}^{\alpha'}.\end{aligned}$$

After some calculations we obtain:

$$(\rho,\eta)\overset{*}{\Gamma}_{b'\alpha'} = M_{b'}^{b}\circ\overset{*}{\pi}\left(-\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot\frac{\partial M_{b}^{\alpha'}\circ\overset{*}{\pi}}{\partial x^{i}}p_{\alpha'} + (\rho,\eta)\overset{*}{\Gamma}_{b\alpha}\right)\Lambda_{\alpha'}^{\alpha}\circ h\circ\overset{*}{\pi}. \qquad \textit{q.e.d.}$$

Remark 4.1.1 If we have a set of real local functions $(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}$ which satisfies the relations of passing (4.1.8), then we have a (ρ,η)-connection $(\rho,\eta)\overset{*}{\Gamma}$ for the fiber bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

Example 4.1.1 If $\overset{*}{\Gamma}$ is a Ehresmann connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ on components $\overset{*}{\Gamma}_{bk}$, then the differentiable real local functions

$$\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\gamma}=\left(\rho_{\gamma}^{k}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\Gamma}_{bk}$$

are the components of a (ρ,η)-connection $\left(\rho,\eta\right)\overset{*}{\Gamma}$ for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ which will be called the *(ρ,η)-connection associated to the Ehresmann connection* $\overset{*}{\Gamma}$.

Definition 4.1.3 If $\left(\rho,\eta\right)\overset{*}{\Gamma}$ is a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then for any

$$z=z^{\alpha}t_{\alpha}\in\Gamma\left(F,\nu,N\right)$$

the application

$$\begin{array}{ccc}\Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right) & \xrightarrow{(\rho,\eta)D_z} & \Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right)\\ \overset{*}{u}=u_{a}s^{a} & \longmapsto & \left(\rho,\eta\right)D_{z}\overset{*}{u}\end{array}$$

defined by

$$\left(\rho,\eta\right)D_{z}\overset{*}{u}=z^{\alpha}\circ h\left(\rho_{\alpha}^{i}\circ h\tfrac{\partial u_{b}}{\partial x^{i}}-\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}\circ\overset{*}{u}\right)s^{b},\tag{4.1.9}$$

will be called the *covariant (ρ,η)-derivative associated to (ρ,η)-connection $\left(\rho,\eta\right)\overset{*}{\Gamma}$ with respect to section z.*

If $h=Id_{M}$ and $\eta=Id_{M}$, then we obtain the *covariant ρ-derivative associated to ρ-connection $\rho\overset{*}{\Gamma}$ with respect to section z.*

In addition, if $\rho=Id_{TM}$, then we obtain the *covariant derivative associated to connection $\overset{*}{\Gamma}$ with respect to the vector field z.*

Definition 4.1.4 We will say that the *(ρ,η)-connection $\left(\rho,\eta\right)\overset{*}{\Gamma}$ is homogeneous* or *linear* if the local real functions $\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\gamma}$ are homogeneous or linear on the fibre of vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ respectively.

Remark 4.1.2 If $\left(\rho,\eta\right)\overset{*}{\Gamma}$ is a linear (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then for each local vector $(m+r)$-chart $\left(U,\overset{*}{s}_{U}\right)$ and for each local vector $(n+p)$-chart (V,t_{V}) such that $U\cap h^{-1}\left(V\right)\neq\phi$, there exists the differentiable real functions $\rho\Gamma_{b\gamma}^{a}$ defined on $U\cap h^{-1}\left(V\right)$ such that

$$\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\gamma}\circ\overset{*}{u}=\left(\rho,\eta\right)\Gamma_{b\gamma}^{a}\cdot u_{a},\tag{4.1.10}$$

for any $\overset{*}{u}=u_{a}s^{a}\in\Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

The differentiable real local functions $\left(\rho,\eta\right)\Gamma_{b\alpha}^{a}$ will be called the *Christoffel coefficients of linear (ρ,η)-connection $\left(\rho,\eta\right)\Gamma$.*

Theorem 4.1.2 *If* $(\rho,\eta)\,\Gamma$ *is a linear* (ρ,η)*-connection for the vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, *then its components satisfy the law of transformation*

$$(4.1.11)\qquad (\rho,\eta)\,\Gamma_{b'\gamma'}^{a'}=M_{b'}^{b}\left[-\rho_{\gamma}^{i}\circ h\frac{\partial M_{b}^{a'}}{\partial x^{i}}+(\rho,\eta)\,\Gamma_{b\gamma}^{a}M_{a}^{a'}\right]\Lambda_{\gamma'}^{\gamma}\circ h.$$

In particular, if $(\rho,\eta)=(Id_{TM},Id_{M})$ *and* $h=Id_{M}$, *then the relations* (4.1.11) *become*

$$(4.1.11')\qquad \Gamma_{j'k'}^{i'}=\frac{\partial x^{j}}{\partial x^{j'}}\left[-\frac{\partial}{\partial x^{i}}\left(\frac{\partial x^{i'}}{\partial x^{j}}\right)+\Gamma_{jk}^{i}\frac{\partial x^{i'}}{\partial x^{i}}\right]\frac{\partial x^{k}}{\partial x^{k'}}.$$

Remark 4.1.3 Since

$$\frac{\partial M_{b}^{a'}}{\partial x^{i}}M_{b'}^{b}+\frac{\partial M_{b'}^{b}}{\partial x^{i}}M_{b}^{a'}=0,$$

it results that the relations (4.1.11) are equivalent with the relations $(4.10')$.

Definition 4.1.5 If $(\rho,\eta)\,\Gamma$ is a linear (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then for any

$$z=z^{\alpha}t_{\alpha}\in\Gamma\left(F,\nu,N\right)$$

the application

$$\begin{array}{ccc}\Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right) & \xrightarrow{(\rho,\eta)D_{z}} & \Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right)\\ \overset{*}{u}=u_{a}s^{a} & \longmapsto & (\rho,\eta)D_{z}\overset{*}{u}\end{array}$$

defined by

$$(4.1.12)\qquad (\rho,\eta)D_{z}\overset{*}{u}=z^{\alpha}\circ h\left(\rho_{\alpha}^{i}\circ h\frac{\partial u_{b}}{\partial x^{i}}-(\rho,\eta)\Gamma_{b\alpha}^{a}\cdot u_{a}\right)s^{b}$$

will be called the *covariant* (ρ,η)*-derivative associated to linear* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$ *with respect to section* z.

If $h=Id_{M}$ and $\eta=Id_{M}$, then we obtain the *covariant* ρ*-derivative associated to linear* ρ*-connection* $\rho\Gamma$ *with respect to section* z.

In addition, if $\rho=Id_{TM}$, then we obtain the *covariant derivative associated to linear connection* Γ *with respect to vector field* z.

In the next we use the same notation $(\rho,\eta)\,\Gamma$ for the linear (ρ,η)-connection for the vector bundle (E,π,M) or for its dual $\left(\overset{*}{E},\overset{*}{\pi},M\right)$

Remark 4.1.4 If $(\rho,\eta)\,\Gamma$ is a linear (ρ,η)-connection for the vector bundle (E,π,M) or for its dual $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ then, the tensor fields algebra

$$\left(\mathcal{T}\left(E,\pi,M\right),+,\cdot,\otimes\right)$$

is endowed with the (ρ,η)-derivative

$$\begin{array}{ccc}\Gamma\left(F,\nu,N\right)\times\mathcal{T}\left(E,\pi,M\right) & \xrightarrow{(\rho,\eta)D} & \mathcal{T}\left(E,\pi,M\right)\\ (z,T) & \longmapsto & (\rho,\eta)\,D_{z}T\end{array}$$

defined for a tensor field $T\in\mathcal{T}_q^p(E,\pi,M)$ by the relation:

$$
\begin{array}{ll}
 & (\rho,\eta)\,D_zT\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)=\Gamma(\rho,\eta)(z)\left(T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)\right)\\
(4.1.13) & -T\left((\rho,\eta)\,D_z\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)-...-T\left(\overset{*}{u}_1,...,(\rho,\eta)\,D_z\overset{*}{u}_p,u_1,...,u_q\right)\\
 & -T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,(\rho,\eta)\,D_zu_1,...,u_q\right)-...-T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,(\rho,\eta)\,D_zu_q\right).
\end{array}
$$

Moreover, it satisfies the condition

$$
(4.1.14)\qquad (\rho,\eta)\,D_{f_1z_1+f_2z_2}T=f_1\,(\rho,\eta)\,D_{z_1}T+f_2\,(\rho,\eta)\,D_{z_2}T.
$$

Consequently, if the tensor algebra $(\mathcal{T}(E,\pi,M),+,\cdot,\otimes)$ is endowed with a (ρ,η)-derivative defined for a tensor field $T\in\mathcal{T}_q^p(E,\pi,M)$ by (4.1.13) which satisfies the condition (4.1.14), then we can endowed (E,π,M) with a linear (ρ,η)-connection $(\rho,\eta)\,\Gamma$ such that its components are defined by the equality:

$$
(\rho,\eta)\,D_{t_\alpha}s_b=(\rho,\eta)\,\Gamma_{b\alpha}^as_a
$$

or

$$
(\rho,\eta)\,D_{t_\alpha}s^a=-\,(\rho,\eta)\,\Gamma_{b\alpha}^as^b.
$$

The (ρ,η)-derivative defined by (4.1.13) will be called the *covariant* (ρ,η)*-derivative.*

After some calculations, we obtain:

$$
\begin{array}{ll}
 & (\rho,\eta)\,D_z\left(T_{b_1,...,b_q}^{a_1,...,a_p}s_{a_1}\otimes...\otimes s_{a_p}\otimes s^{b_1}\otimes...\otimes s^{b_q}\right)\\
 & =z^\alpha\circ h\left(\rho_\alpha^i\circ h\dfrac{\partial T_{b_1,...,b_q}^{a_1,...,a_p}}{\partial x^i}+(\rho,\eta)\,\Gamma_{a\alpha}^{a_1}T_{b_1,...,b_q}^{a,a_2...,a_p}\right.\\
(4.1.15) & +(\rho,\eta)\,\Gamma_{a\alpha}^{a_2}T_{b_1,...,b_q}^{a_1,a...,a_p}+...+(\rho,\eta)\,\Gamma_{a\alpha}^{a_p}T_{b_1,...,b_q}^{a_1,a_2...,a}-...\\
 & -(\rho,\eta)\,\Gamma_{b_1\alpha}^{b}T_{b,b_2,...,b_q}^{a_1,a_2...,a_p}-(\rho,\eta)\,\Gamma_{b_2\alpha}^{b}T_{b_1,b,...,b_q}^{a_1,a_2...,a_p}-...\\
 & \left.-(\rho,\eta)\,\Gamma_{b_q\alpha}^{b}T_{b_1,b_2,...,b}^{a_1,a_2...,a_p}\right)s_{a_1}\otimes...\otimes s_{a_p}\otimes s^{b_1}\otimes...\otimes s^{b_q}\\
 & \overset{put}{=}z^\alpha\circ hT_{b_1,...,b_q|\alpha}^{a_1,...,a_p}s_{a_1}\otimes...\otimes s_{a_p}\otimes s^{b_1}\otimes...\otimes s^{b_q}.
\end{array}
$$

If $(\rho,\eta)\,\Gamma$ is the linear (ρ,η)-connection associated to the Ehresmann linear connection Γ, namely $(\rho,\eta)\,\Gamma_{b\alpha}^a=\left(\rho_\alpha^k\circ h\right)\Gamma_{bk}^a$, then

$$
(4.1.16)\qquad T_{b_1,...,b_q|\alpha}^{a_1,...,a_p}=\left(\rho_\alpha^k\circ h\right)T_{b_1,...,b_q|k}^{a_1,...,a_p}.
$$

In particular, if $h=Id_M$, then we obtain the formula:

$$
\begin{array}{ll}
 & (\rho,\eta)\,D_z\left(T_{b_1,...,b_q}^{a_1,...,a_p}s_{a_1}\otimes...\otimes s_{a_p}\otimes s^{b_1}\otimes...\otimes s^{b_q}\right)\\
 & =z^\alpha\left(\rho_\alpha^i\dfrac{\partial T_{b_1,...,b_q}^{a_1,...,a_p}}{\partial x^i}+(\rho,\eta)\,\Gamma_{a\alpha}^{a_1}T_{b_1,...,b_q}^{a,a_2...,a_p}\right.\\
(4.1.17) & +(\rho,\eta)\,\Gamma_{a\alpha}^{a_2}T_{b_1,...,b_q}^{a_1,a...,a_p}+...+(\rho,\eta)\,\Gamma_{a\alpha}^{a_p}T_{b_1,...,b_q}^{a_1,a_2...,a}-...\\
 & -(\rho,\eta)\,\Gamma_{b_1\alpha}^{b}T_{b,b_2,...,b_q}^{a_1,a_2...,a_p}-(\rho,\eta)\,\Gamma_{b_2\alpha}^{b}T_{b_1,b,...,b_q}^{a_1,a_2...,a_p}-...\\
 & \left.-(\rho,\eta)\,\Gamma_{b_q\alpha}^{b}T_{b_1,b_2,...,b}^{a_1,a_2...,a_p}\right)s_{a_1}\otimes...\otimes s_{a_p}\otimes s^{b_1}\otimes...\otimes s^{b_q}\\
 & \overset{put}{=}z^\alpha T_{b_1,...,b_q|\alpha}^{a_1,...,a_p}s_{a_1}\otimes...\otimes s_{a_p}\otimes s^{b_1}\otimes...\otimes s^{b_q}.
\end{array}
$$

5 Torsion and curvature. Formulas of Ricci and Bianchi type

We apply our theory for the diagram:

$$
\begin{array}{ccc}
E & & \left(F,[,]_{F,h},(\rho,Id_N)\right) \\
\pi\downarrow & & \downarrow\nu \\
M & \xrightarrow{h} & N
\end{array}
\tag{5.1}
$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,Id_N)\right)\in|\mathbf{GLA}|$.

Let $\rho\Gamma$ be a linear ρ-connection for the vector bundle (E,π,M) by components $\rho\Gamma_{b\alpha}^{a}$.

Using the components of the linear ρ-connection $\rho\Gamma$, then we obtain a linear ρ-connection $\rho\dot{\Gamma}$ for the vector bundle (E,π,M) given by the diagram:

$$
\begin{array}{ccc}
E & & \left(h^*F,[,]_{h^*F},\left(\overset{h^*F}{\rho},Id_M\right)\right) \\
\pi\downarrow & & \downarrow h^*\nu \\
M & \xrightarrow{Id_M} & M
\end{array}
\tag{5.2}
$$

If $(E,\pi,M)=(F,\nu,N)$, then, using the components of the linear ρ-connection $\rho\Gamma$, we can consider a linear ρ-connection $\rho\ddot{\Gamma}$ for the vector bundle $(h^*E,h^*\pi,M)$ given by the diagram:

$$
\begin{array}{ccc}
h^*E & & \left(h^*E,[,]_{h^*E},\left(\overset{h^*E}{\rho},Id_M\right)\right) \\
h^*\pi\downarrow & & \downarrow h^*\pi \\
M & \xrightarrow{Id_M} & M
\end{array}
\tag{5.3}
$$

Definition 5.1 If $(E,\pi,M)=(F,\nu,N)$, then the application

$$
\begin{array}{ccc}
\Gamma(h^*E,h^*\pi,M)^2 & \xrightarrow{(\rho,h)\mathbb{T}} & \Gamma(h^*E,h^*\pi,M) \\
(U,V) & \longrightarrow & \rho\mathbb{T}(U,V)
\end{array}
$$

defined by:

$$
(\rho,h)\,\mathbb{T}(U,V)=\rho\ddot{D}_UV-\rho\ddot{D}_VU-[U,V]_{h^*E}\,,
\tag{5.4}
$$

for any $U,V\in\Gamma(h^*E,h^*\pi,M)$, will be called *$(\rho,h)$-torsion associated to linear ρ-connection $\rho\Gamma$.*

Remark 5.1 In particular, if $h=Id_M$, then we obtain the application

$$
\begin{array}{ccc}
\Gamma(E,\pi,M)^2 & \xrightarrow{\rho\mathbb{T}} & \Gamma(E,\pi,M) \\
(u,v) & \longrightarrow & \rho\mathbb{T}(u,v)
\end{array}
$$

defined by:

$$
\rho\mathbb{T}(u,v)=\rho D_uv-\rho D_vu-[u,v]_E\,,
\tag{5.4'}
$$

for any $u, v \in \Gamma(E, \pi, M)$, which will be called *$\rho$-torsion associated to linear ρ-connection $\rho\Gamma$*.

Moreover, if $\rho = Id_{TM}$, then we obtain the torsion $\mathbb{T}$ associated to the linear connection Γ.

Proposition 5.1 *The (ρ, h)-torsion $(\rho, h)\,\mathbb{T}$ associated to the linear ρ-connection $\rho\Gamma$ is $\mathbb{R}$-bilinear and antisymmetric.*

If

$$(\rho, h)\,\mathbb{T}(S_a, S_b) \overset{put}{=} (\rho, h)\,\mathbb{T}^c{}_{ab} S_c$$

then

$$(\rho, h)\,\mathbb{T}^c{}_{ab} = \rho\Gamma^c_{ab} - \rho\Gamma^c_{ba} - L^c_{ab} \circ h. \tag{5.5}$$

In particular, if $h = Id_M$ and $\rho\mathbb{T}(s_a, s_b) \overset{put}{=} \rho\mathbb{T}^c_{ab} s_c$, then

$$\rho\mathbb{T}^c{}_{ab} = \rho\Gamma^c_{ab} - \rho\Gamma^c_{ba} - L^c_{ab}. \tag{5.5'}$$

Moreover, if $\rho = Id_{TM}$, then the equality $(5.5')$ becomes:

$$\mathbb{T}^i{}_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}. \tag{5.5''}$$

Definition 5.2 The application

$$\begin{array}{ccc} (\Gamma(h^*F, h^*\nu, M)^2 \times \Gamma(E, \pi, M) & \xrightarrow{(\rho,h)\mathbb{R}} & \Gamma(E, \pi, M) \\ ((Z, V), u) & \longrightarrow & \rho\mathbb{R}(Z, V)u \end{array}$$

defined by

$$(\rho, h)\,\mathbb{R}(Z, V)\,u = \rho\dot{D}_Z\left(\rho\dot{D}_V u\right) - \rho\dot{D}_V\left(\rho\dot{D}_Z u\right) - \rho\dot{D}_{[Z,V]_{h^*F}} u, \tag{5.6}$$

for any $Z, V \in \Gamma(h^*F, h^*\nu, M)$ and $u \in \Gamma(E, \pi, M)$, will be called *(ρ, h)-curvature associated to the linear ρ-connection $\rho\Gamma$*.

Remark 5.1 In particular, if $h = Id_M$, then we obtain the application

$$\begin{array}{ccc} \Gamma(F, \nu, M)^2 \times \Gamma(E, \pi, M) & \xrightarrow{\rho\mathbb{R}} & \Gamma(E, \pi, M) \\ ((z, v), u) & \longrightarrow & \rho\mathbb{R}(z, v)u \end{array}$$

defined by

$$\rho\mathbb{R}(z, v)\,u = \rho D_z(\rho D_v u) - \rho D_v(\rho D_z u) - \rho D_{[z,v]_F} u, \tag{5.6'}$$

for any $z, v \in \Gamma(F, \nu, M)$ and $u \in \Gamma(E, \pi, M)$, which will be called *ρ-curvature associated to the linear ρ-connection $\rho\Gamma$*.

Moreover, if $\rho = Id_{TM}$, then we obtain the curvature $\mathbb{R}$ associated to the linear connection Γ.

Proposition 5.2 *The (ρ, h)-curvature $(\rho, h)\,\mathbb{R}$ associated to the linear ρ-connection $\rho\Gamma$, is $\mathbb{R}$-linear in each argument and antisymmetric in the first two arguments.*

If

$$(\rho, h)\,\mathbb{R}\,(T_\beta, T_\alpha)\,s_b \overset{put}{=} (\rho, h)\,\mathbb{R}_{b\ \alpha\beta}^{a} s_a,$$

then

$$(5.7)\qquad \begin{array}{rl} (\rho, h)\,\mathbb{R}_{b\ \alpha\beta}^{a} & = \rho_\beta^j \circ h\frac{\partial \rho\Gamma_{b\alpha}^a}{\partial x^j} + \rho\Gamma_{e\beta}^a \rho\Gamma_{b\alpha}^e - \rho_\alpha^i \circ h\frac{\partial \rho\Gamma_{b\beta}^a}{\partial x^i} \\ & -\rho\Gamma_{e\alpha}^a \rho\Gamma_{b\beta}^e + \rho\Gamma_{b\gamma}^a L_{\alpha\beta}^\gamma \circ h. \end{array}$$

In particular, if $h = Id_M$ *and* $\rho\mathbb{R}\,(t_\beta, t_\alpha)\,s_b \overset{put}{=} \rho\mathbb{R}_{b\ \alpha\beta}^{a} s_a$, *then*

$$(5.7')\qquad \rho\mathbb{R}_{b\ \alpha\beta}^{a} = \rho_\beta^j \frac{\partial \rho\Gamma_{b\alpha}^a}{\partial x^j} + \rho\Gamma_{e\beta}^a \rho\Gamma_{b\alpha}^e - \rho_\alpha^i \frac{\partial \rho\Gamma_{b\beta}^a}{\partial x^i} - \rho\Gamma_{e\alpha}^a \rho\Gamma_{b\beta}^e + \rho\Gamma_{b\gamma}^a L_{\alpha\beta}^\gamma.$$

Moreover, if $\rho = Id_{TM}$, *then equality* $(5.7')$ *becomes:*

$$(5.7'')\qquad \mathbb{R}_{b\ hk}^{a} = \frac{\partial \Gamma_{bh}^a}{\partial x^k} + \Gamma_{ek}^a \Gamma_{bh}^e - \frac{\partial \Gamma_{bk}^a}{\partial x^h} - \Gamma_{eh}^a \Gamma_{bk}^e.$$

Theorem 5.1 *For any* $u^a s_a \in \Gamma\,(E, \pi, M)$ *we shall use the notation*

$$(5.8)\qquad u^a{}_{|\alpha\beta} = \rho_\beta^j \circ h\frac{\partial}{\partial x^j}\left(u^{a_1}{}_{|\alpha}\right) + \rho\Gamma_{b_\beta}^{a_1} u^b{}_{|\alpha},$$

and we verify the formulas:

$$(5.9)\qquad u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} = u^a\,(\rho, h)\,\mathbb{R}_{a\ \alpha\beta}^{a_1} - u^{a_1}{}_{|\gamma} L_{\alpha\beta}^\gamma \circ h.$$

After some calculations, we obtain

$$(5.10)\qquad (\rho, h)\,\mathbb{R}_{a\ \alpha\beta}^{a_1} = u_a\left(u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} + u^{a_1}{}_{|\gamma} L_{\alpha\beta}^\gamma \circ h\right),$$

where $u_a s^a \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ *such that* $u_a u^b = \delta_a^b$.

In particular, if $h = Id_M$, *then the relations* (5.10) *become*

$$(5.10')\qquad \rho\mathbb{R}_{a\ \alpha\beta}^{a_1} = u_a\left(u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} + u^{a_1}{}_{|\gamma} L_{\alpha\beta}^\gamma\right).$$

Moreover, if $\rho = id_{TM}$, *then the relations* $(5.10')$ *become*

$$(5.10'')\qquad \mathbb{R}_{a\ ij}^{a_1} = u_a\left(u^{a_1}{}_{|ij} - u^{a_1}{}_{|ji}\right).$$

Proof. Since

$$\begin{array}{rl} u^{a_1}{}_{|\alpha\beta} = & \rho_\beta^j \circ h\left(\dfrac{\partial}{\partial x^j}\left(\rho_\alpha^i \circ h\dfrac{\partial u^{a_1}}{\partial x^i} + \rho\Gamma_{a\alpha}^{a_1} u^a\right)\right) \\ & + \rho\Gamma_{b\beta}^{a_1}\left(\rho_\alpha^i \circ h\dfrac{\partial u^b}{\partial x^i} + \rho\Gamma_{a\alpha}^b u^a\right) \\ & = \rho_\beta^j \circ h\dfrac{\partial \rho_\alpha^i \circ h}{\partial x^j}\dfrac{\partial u^{a_1}}{\partial x^i} + \rho_\beta^j \circ h\rho_\alpha^i \circ h\dfrac{\partial}{\partial x^j}\left(\dfrac{\partial u^{a_1}}{\partial x^i}\right) \\ & + \rho_\beta^j \circ h\dfrac{\partial \rho\Gamma_{a\alpha}^{a_1}}{\partial x^j} u^a + \rho_\beta^j \circ h\rho\Gamma_{a\alpha}^{a_1}\dfrac{\partial u^a}{\partial x^j} \\ & + \rho_\alpha^i \circ h\rho\Gamma_{b\beta}^{a_1}\dfrac{\partial u^b}{\partial x^i} + \rho\Gamma_{b\beta}^{a_1}\rho\Gamma_{a\alpha}^b u^a \end{array}$$

and

$$\begin{aligned}u^{a_1}{}_{|\beta\alpha} &= \rho_\alpha^i\circ h\left(\frac{\partial}{\partial x^i}\left(\rho_\beta^j\circ h\frac{\partial u^{a_1}}{\partial x^j}+\rho\Gamma_{a\beta}^{a_1}u^a\right)\right)\\
&+\rho\Gamma_{b\alpha}^{a_1}\left(\rho_\beta^j\circ h\frac{\partial u^b}{\partial x^j}+\rho\Gamma_{a\beta}^{b}u^a\right)\\
&=\rho_\alpha^i\circ h\frac{\partial\rho_\beta^j\circ h}{\partial x^i}\frac{\partial u^{a_1}}{\partial x^j}+\rho_\beta^j\circ h\rho_\alpha^i\circ h\frac{\partial}{\partial x^i}\left(\frac{\partial u^{a_1}}{\partial x^j}\right)\\
&+\rho_\alpha^i\circ h\frac{\partial\rho\Gamma_{a\beta}^{a_1}}{\partial x^i}u^a+\rho_\alpha^i\circ h\rho\Gamma_{a\beta}^{a_1}\frac{\partial u^a}{\partial x^i}\\
&+\rho_\beta^j\circ h\rho\Gamma_{b\alpha}^{a_1}\frac{\partial u^b}{\partial x^j}+\rho\Gamma_{b\alpha}^{a_1}\rho\Gamma_{a\beta}^{b}u^a,\end{aligned}$$

it results that

$$\begin{aligned}u^{a_1}{}_{|\alpha\beta}-u^{a_1}{}_{|\beta\alpha} &= \rho_\beta^j\circ h\frac{\partial\rho_\alpha^i\circ h}{\partial x^j}\frac{\partial u^{a_1}}{\partial x^i}-\rho_\alpha^i\circ h\frac{\partial\rho_\beta^j\circ h}{\partial x^i}\frac{\partial u^{a_1}}{\partial x^j}\\
&+\left(\rho_\beta^j\circ h\rho_\alpha^i\circ h\frac{\partial^2u^{a_1}}{\partial x^i\partial x^j}-\rho_\beta^j\circ h\rho_\alpha^i\circ h\frac{\partial^2u^{a_1}}{\partial x^j\partial x^i}\right)\\
&+\left(\rho_\beta^j\circ h\frac{\partial\rho\Gamma_{a\alpha}^{a_1}}{\partial x^j}u^a-\rho_\alpha^i\circ h\frac{\partial\rho\Gamma_{a\beta}^{a_1}}{\partial x^i}u^a\right)\\
&+\left(\rho_\beta^j\circ h\rho\Gamma_{a\alpha}^{a_1}\frac{\partial u^a}{\partial x^j}-\rho_\beta^j\circ h\rho\Gamma_{b\alpha}^{a_1}\frac{\partial u^b}{\partial x^j}\right)\\
&+\left(\rho_\alpha^i\circ h\rho\Gamma_{b\beta}^{a_1}\frac{\partial u^b}{\partial x^i}-\rho_\alpha^i\circ h\rho\Gamma_{a\beta}^{a_1}\frac{\partial u^a}{\partial x^i}\right)\\
&+\rho\Gamma_{b\beta}^{a_1}\rho\Gamma_{a\alpha}^{b}u^a-\rho\Gamma_{b\alpha}^{a_1}\rho\Gamma_{a\beta}^{b}u^a.\end{aligned}$$

After some calculations, we obtain:

$$\begin{aligned}u^{a_1}{}_{|\alpha\beta}-u^{a_1}{}_{|\beta\alpha} &= L_{\beta\alpha}^\gamma\circ h\rho_\gamma^k\circ h\frac{\partial u^{a_1}}{\partial x^k}\\
&+\left(\rho_\beta^j\circ h\frac{\partial\rho\Gamma_{a\alpha}^{a_1}}{\partial x^j}u^a-\rho_\alpha^i\circ h\frac{\partial\rho\Gamma_{a\beta}^{a_1}}{\partial x^i}u^a\right)\\
&+\rho\Gamma_{b\beta}^{a_1}\rho\Gamma_{a\alpha}^{b}u^a-\rho\Gamma_{b\alpha}^{a_1}\rho\Gamma_{a\beta}^{b}u^a.\end{aligned}$$

Since

$$\begin{aligned}u^a\left(\rho,h\right)\mathbb{R}_{a\ \alpha\beta}^{a_1} &= u^a\left(\rho_\beta^j\circ h\frac{\partial\rho\Gamma_{a\alpha}^{a_1}}{\partial x^j}+\rho\Gamma_{e\beta}^{a_1}\rho\Gamma_{a\alpha}^{e}-\rho_\alpha^i\circ h\frac{\partial\rho\Gamma_{a\beta}^{a_1}}{\partial x^i}\right.\\
&\left.-\rho\Gamma_{e\alpha}^{a_1}\rho\Gamma_{a\beta}^{e}-\rho\Gamma_{a\gamma}^{a_1}L_{\beta\alpha}^\gamma\circ h\right).\end{aligned}$$

and

$$u^{a_1}{}_{|\gamma}L_{\alpha\beta}^\gamma\circ h=\left(\rho_\gamma^k\circ h\frac{\partial u^{a_1}}{\partial x^k}+\rho\Gamma_{a\gamma}^{a_1}u^a\right)L_{\alpha\beta}^\gamma\circ h$$

it results that

$$\begin{aligned} u^{a}\left(\rho,h\right)\mathbb{R}_{a\ \alpha\beta}^{a_{1}}-u^{a_{1}}{}_{|\gamma}L_{\alpha\beta}^{\gamma}\circ h = & -L_{\alpha\beta}^{\gamma}\circ h\rho_{\gamma}^{k}\circ h\frac{\partial u^{a_{1}}}{\partial x^{k}} \\ & +\left(\rho_{\beta}^{j}\circ h\frac{\partial\rho\Gamma_{a\alpha}^{a_{1}}}{\partial x^{j}}u^{a}-\rho_{\alpha}^{i}\circ h\frac{\partial\rho\Gamma_{a\beta}^{a_{1}}}{\partial x^{i}}u^{a}\right) \\ & +\rho\Gamma_{b\beta}^{a_{1}}\rho\Gamma_{a\alpha}^{b}u^{a}-\rho\Gamma_{b\alpha}^{a_{1}}\rho\Gamma_{a\beta}^{b}u^{a}. \end{aligned}$$

q.e.d.

Lemma 5.1 *If* $(E,\pi,M)=(F,\nu,N)$, *then, for any* $u^{a}s_{a}\in\Gamma\left(E,\pi,M\right)$, *we have that* $u^{a}{}_{|c},\ a,c\in\overline{1,n}$ are the *components of a tensor field of* $(1,1)$ *type.*

Proof. Let U and U' be two vector local $(m+n)$-charts such that $U\cap U'\neq\phi$.
Since $u^{a'}\left(x\right)=M_{a}^{a'}\left(x\right)u^{a}\left(x\right),$ for any $x\in U\cap U'$, it results that

$$\rho_{c'}^{k'}\circ h\left(x\right)\tfrac{\partial u^{a'}(x)}{\partial x^{k'}}=\rho_{c'}^{k'}\circ h\left(x\right)\tfrac{\partial}{\partial x^{k'}}\left(M_{a}^{a'}\left(x\right)\right)u^{a}\left(x\right)+M_{a}^{a'}\left(x\right)\rho_{c'}^{k'}\circ h\left(x\right)\tfrac{\partial u^{a}(x)}{\partial x^{k'}}. \quad (1)$$

Since , for any $x\in U\cap U'$, we have

$$\rho\Gamma_{b'c'}^{a'}(x)=M_{a}^{a'}(x)\left(\rho_{c}^{k}\circ h(x)\tfrac{\partial}{\partial x^{k}}(M_{b'}^{a}(x))+\rho\Gamma_{bc}^{a}(x)M_{b'}^{b}(x)\right)M_{c'}^{c}(x), \quad (2)$$

and

$$0=\tfrac{\partial}{\partial x^{k'}}\left(M_{a}^{a'}\left(x\right)M_{b'}^{a}\left(x\right)\right)=\tfrac{\partial}{\partial x^{k'}}\left(M_{a}^{a'}\left(x\right)\right)M_{b'}^{a}\left(x\right)+M_{a}^{a'}\left(x\right)\tfrac{\partial}{\partial x^{k'}}\left(M_{b'}^{a}\left(x\right)\right) \quad (3)$$

it results that

$$\begin{aligned} \rho\Gamma_{b'c'}^{a'}\left(x\right)u^{b'}\left(x\right) = & -\rho_{c'}^{k'}\circ h\left(x\right)\frac{\partial}{\partial x^{k'}}\left(M_{a}^{a'}\left(x\right)\right)u^{a}\left(x\right) \\ & +M_{a}^{a'}\left(x\right)\rho\Gamma_{bc}^{a}\left(x\right)u^{b}\left(x\right)M_{c'}^{c}\left(x\right). \end{aligned} \quad (4)$$

Summing the equalities (1) and (4), it results the conclusion of lemma. *q.e.d.*

Theorem 5.2 *If* $(E,\pi,M)=(F,\nu,N)$, *then, for any*

$$u^{a}s_{a}\in\Gamma\left(E,\pi,M\right),$$

we shall use the notation

(5.11) $$u^{a_{1}}{}_{|a|b}=u^{a_{1}}{}_{|ab}-\rho\Gamma_{ab}^{d}u^{a_{1}}{}_{|d}$$

and we verify the formulas of Ricci type

(5.12) $$u^{a_{1}}{}_{|a|b}-u^{a_{1}}{}_{|b|a}+\left(\rho,h\right)\mathbb{T}^{d}{}_{ab}u^{a_{1}}{}_{|d}=u^{d}\left(\rho,h\right)\mathbb{R}_{d\ ab}^{a_{1}}-u^{a_{1}}{}_{|c}L_{ab}^{c}\circ h$$

In particular, if $h=Id_{M}$, *then the relations* (5.12) *become*

(5.12′) $$u^{a_{1}}{}_{|a|b}-u^{a_{1}}{}_{|b|a}+\rho\mathbb{T}^{d}{}_{ab}u^{a_{1}}{}_{|d}=u^{d}\rho\mathbb{R}_{d\ ab}^{a_{1}}-u^{a_{1}}{}_{|c}L_{ab}^{c}$$

Moreover, if $\rho = id_{TM}$, *then the relations* (5.12′) *become*

$$u^{i_1}{}_{|i|j} - u^{i_1}{}_{|i|j} + \mathbb{T}^{k}{}_{ij} u^{i_1}{}_{|k} = u^k \mathbb{R}^{i_1}_{k\ ij} \tag{5.12″}$$

Theorem 5.3 *For any* $u_a s^a \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ *we shall use the notation*

$$u_{b_1|\alpha\beta} = \rho^j_\beta \circ h \tfrac{\partial}{\partial x^j}\left(u_{b_1|\alpha}\right) - \rho\Gamma^b_{b_1\beta} u_{b|\alpha} \tag{5.13}$$

and we verify the formulas:

$$u_{b_1|\alpha\beta} - u_{b_1|\beta\alpha} = -u_b\left(\rho, h\right)\mathbb{R}^b_{b_1\ \alpha\beta} - u_{b_1|\gamma} L^\gamma_{\alpha\beta} \circ h \tag{5.14}$$

After some calculations, we obtain

$$\left(\rho, h\right)\mathbb{R}^b_{b_1\ \alpha\beta} = u^b\left(-u_{b_1|\alpha\beta} + u_{b_1|\beta\alpha} - u_{b_1|\gamma} L^\gamma_{\alpha\beta} \circ h\right), \tag{5.15}$$

where $u^a s_a \in \Gamma\left(E, \pi, M\right)$ *such that* $u_a u^b = \delta^b_a$.

In particular, if $h = Id_M$, *then the relations* (5.15) *become*

$$\rho\mathbb{R}^b_{b_1\ \alpha\beta} = u^b\left(-u_{b_1|\alpha\beta} + u_{b_1|\beta\alpha} - u_{b_1|\gamma} L^\gamma_{\alpha\beta}\right). \tag{5.15′}$$

Moreover, if $\rho = id_{TM}$ *then the relations* (5.15′) *become*

$$\mathbb{R}^b_{b_1\ ij} = u^b\left(-u_{b_1|ij} + u_{b_1|ji}\right). \tag{5.15″}$$

Proof. Since

$$\begin{aligned}
u_{b_1|\alpha\beta} &= \rho^j_\beta \circ h\left(\frac{\partial}{\partial x^j}\left(\rho^i_\alpha \circ h\frac{\partial u_{b_1}}{\partial x^i} - \rho\Gamma^b_{b_1\alpha} u_b\right)\right)\\
&- \rho\Gamma^b_{b_1\beta}\left(\rho^i_\alpha \circ h\frac{\partial u_b}{\partial x^i} - \rho\Gamma^a_{b\alpha} u_a\right)\\
&= \rho^j_\beta \circ h\frac{\partial \rho^i_\alpha \circ h}{\partial x^j}\frac{\partial u_{b_1}}{\partial x^i} + \rho^j_\beta \circ h\rho^i_\alpha \circ h\frac{\partial}{\partial x^j}\left(\frac{\partial u_{b_1}}{\partial x^i}\right)\\
&- \rho^j_\beta \circ h\frac{\partial \rho\Gamma^b_{b_1\alpha}}{\partial x^j} u_b - \rho^j_\beta \circ h\rho\Gamma^b_{b_1\alpha}\frac{\partial u_b}{\partial x^j}\\
&- \rho^i_\alpha \circ h\rho\Gamma^b_{b_1\beta}\frac{\partial u_b}{\partial x^i} + \rho\Gamma^b_{b_1\beta}\rho\Gamma^a_{b\alpha} u_a
\end{aligned}$$

and

$$\begin{aligned}
u_{b_1|\beta\alpha} &= \rho^i_\alpha \circ h\left(\frac{\partial}{\partial x^i}\left(\rho^j_\beta \circ h\frac{\partial u_{b_1}}{\partial x^j} - \rho\Gamma^b_{b_1\beta} u_b\right)\right)\\
&- \rho\Gamma^b_{b_1\alpha}\left(\rho^j_\beta \circ h\frac{\partial u_b}{\partial x^j} - \rho\Gamma^a_{b\beta} u_a\right)\\
&= \rho^i_\alpha \circ h\frac{\partial \rho^j_\beta \circ h}{\partial x^i}\frac{\partial u_{b_1}}{\partial x^i} + \rho^j_\beta \circ h\rho^i_\alpha \circ h\frac{\partial}{\partial x^i}\left(\frac{\partial u_{b_1}}{\partial x^j}\right)\\
&- \rho^i_\alpha \circ h\frac{\partial \rho\Gamma^b_{b_1\beta}}{\partial x^i} u_b - \rho^i_\alpha \circ h\rho\Gamma^b_{b_1\beta}\frac{\partial u_b}{\partial x^i}\\
&- \rho^j_\beta \circ h\rho\Gamma^b_{b_1\alpha}\frac{\partial u_b}{\partial x^i} + \rho\Gamma^b_{b_1\alpha}\rho\Gamma^a_{b\beta} u_a
\end{aligned}$$

it results that

$$
\begin{aligned}
u_{b_1|\alpha\beta} - u_{b_1|\beta\alpha} = {} & \rho_\beta^j \circ h \frac{\partial \rho_\alpha^i \circ h}{\partial x^j}\frac{\partial u_{b_1}}{\partial x^i} - \rho_\alpha^i \circ h \frac{\partial \rho_\beta^j \circ h}{\partial x^i}\frac{\partial u_{b_1}}{\partial x^j} \\
& + \rho_\beta^j \circ h \rho_\alpha^i \circ h \frac{\partial}{\partial x^j}\left(\frac{\partial u_{b_1}}{\partial x^i}\right) - \rho_\beta^j \circ h \rho_\alpha^i \circ h \frac{\partial}{\partial x^i}\left(\frac{\partial u_{b_1}}{\partial x^j}\right) \\
& + \rho_\alpha^i \circ h \frac{\partial \rho\Gamma_{b_1\beta}^b}{\partial x^i} u_b - \rho_\beta^j \circ h \frac{\partial \rho\Gamma_{b_1\alpha}^b}{\partial x^j} u_b \\
& + \rho_\beta^j \circ h \rho\Gamma_{b_1\alpha}^b \frac{\partial u_b}{\partial x^j} - \rho_\beta^j \circ h \rho\Gamma_{b_1\alpha}^b \frac{\partial u_b}{\partial x^j} \\
& + \rho_\alpha^i \circ h \rho\Gamma_{b_1\alpha}^b \frac{\partial u_b}{\partial x^i} - \rho_\alpha^i \circ h \rho\Gamma_{b_1\alpha}^b \frac{\partial u_b}{\partial x^i} \\
& + \rho\Gamma_{b_1\beta}^b \rho\Gamma_{b\alpha}^a u_a - \rho\Gamma_{b_1\alpha}^b \rho\Gamma_{b\beta}^a u_a.
\end{aligned}
$$

After some calculations, we obtain:

$$
\begin{aligned}
u_{b_1|\alpha\beta} - u_{b_1|\beta\alpha} & = L_{\beta\alpha}^\gamma \circ h \rho_\gamma^k \circ h \tfrac{\partial u_{b_1}}{\partial x^k} \\
& + \left(\rho_\alpha^i \circ h \tfrac{\partial \rho\Gamma_{b_1\beta}^b}{\partial x^i} u_b - \rho_\beta^j \circ h \tfrac{\partial \rho\Gamma_{b_1\alpha}^b}{\partial x^j} u_b\right) \\
& + \rho\Gamma_{b_1\beta}^b \rho\Gamma_{b\alpha}^a u_a - \rho\Gamma_{b_1\alpha}^b \rho\Gamma_{b\beta}^a u_a.
\end{aligned}
$$

Since

$$
\begin{aligned}
u_b(\rho, h)\,\mathbb{R}_{b_1\alpha\beta}^b = u_b \Bigg(& \rho_\beta^j \circ h \frac{\partial \rho\Gamma_{b_1\alpha}^b}{\partial x^j} + \rho\Gamma_{e\beta}^b \rho\Gamma_{b_1\alpha}^e \\
& - \rho_\alpha^i \circ h \frac{\partial \rho\Gamma_{b_1\beta}^b}{\partial x^i} - \rho\Gamma_{e\alpha}^b \rho\Gamma_{b_1\beta}^e - \rho\Gamma_{b_1\gamma}^b L_{\beta\alpha}^\gamma \circ h \Bigg)
\end{aligned}
$$

and

$$
u_{b_1|\gamma} L_{\alpha\beta}^\gamma \circ h = \left(\rho_\gamma^k \circ h \frac{\partial u_{b_1}}{\partial x^k} - \rho\Gamma_{b_1\gamma}^b u_b\right) L_{\alpha\beta}^\gamma \circ h
$$

it results that

$$
\begin{aligned}
-u_b(\rho, h)\,\mathbb{R}_{b_1,\alpha\beta}^b - u_{b_1|\gamma} L_{\alpha\beta}^\gamma \circ h = {} & -L_{\alpha\beta}^\gamma \circ h \rho_\gamma^k \circ h \frac{\partial u_{b_1}}{\partial x^k} \\
& + \left(\rho_\alpha^i \circ h \frac{\partial \rho\Gamma_{b_1\beta}^b}{\partial x^i} u_b - \rho_\beta^j \circ h \frac{\partial \rho\Gamma_{b_1\alpha}^b}{\partial x^j} u_b\right) \\
& + \rho\Gamma_{b_1\beta}^b \rho\Gamma_{b\alpha}^a u_a - \rho\Gamma_{b_1\alpha}^b \rho\Gamma_{b\beta}^a u_a.
\end{aligned}
$$

q.e.d.

Lemma 5.2 *If* $(E, \pi, M) = (F, \nu, N)$*, then, for any*

$$
u_b s^b \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right),
$$

we have that $u_{b\ |c}$*,* $b, c \in \overline{1, n}$ are the *components of a tensor field of* $(0, 2)$ *type.*

Proof. Let U and U' be two vector local $(m+n)$-charts such that $U \cap U' \neq \phi$.

Since $u_{b'}(x) = M_{b'}^{b}(x)\, u_b(x)$, for any $x \in U \cap U'$, it results that

$$\begin{aligned}\rho_{c'}^{k'} \circ h(x) \frac{\partial u_{b'}(x)}{\partial x^{k'}} &= \rho_{c'}^{k'} \circ h(x) \frac{\partial}{\partial x^{k'}} \left(M_{b'}^{b}(x) \right) u_b(x) \\ &+ M_{b'}^{b}(x)\, \rho_{c'}^{k'} \circ h(x) \frac{\partial u_b(x)}{\partial x^{k'}}.\end{aligned} \tag{1}$$

Since, for any $x \in U \cap U'$, we have

$$\begin{aligned}\rho\Gamma_{b'c'}^{a'}(x) &= M_{a}^{a'}(x) \left(\rho_{c}^{k} \circ h(x) \frac{\partial}{\partial x^{k}} \left(M_{b'}^{a}(x) \right) \right. \\ &\left. + \rho\Gamma_{bc}^{a}(x)\, M_{b'}^{b}(x) \right) M_{c'}^{c}(x),\end{aligned} \tag{2}$$

and

$$\begin{aligned}0 &= \frac{\partial}{\partial x^{k'}} \left(M_{a}^{a'}(x)\, M_{b'}^{a}(x) \right) \\ &= \frac{\partial}{\partial x^{k'}} \left(M_{a}^{a'}(x) \right) M_{b'}^{a}(x) + M_{a}^{a'}(x) \frac{\partial}{\partial x^{k'}} \left(M_{b'}^{a}(x) \right)\end{aligned} \tag{3}$$

it results that

$$\begin{aligned}\rho\Gamma_{b'c'}^{a'}(x)\, u_{a'}(x) = &-\rho_{c'}^{k'} \circ h(x) \frac{\partial}{\partial x^{k'}} \left(M_{b'}^{b}(x) \right) u_b(x) \\ &+ M_{b'}^{b}(x)\, \rho\Gamma_{bc}^{a}(x)\, u_a(x)\, M_{c'}^{c}(x).\end{aligned} \tag{4}$$

Summing the equalities (1) and (4), it results the conclusion of lemma. *q.e.d.*

Theorem 5.4 *If* $(E, \pi, M) = (F, \nu, N)$*, then, for any*

$$u_b s^b \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M \right),$$

we shall use the notation

$$u_{b_1\ |a|b} = u_{b_1\ |ab} - \rho\Gamma_{ab}^{d} u_{b_1\ |d} \tag{5.16}$$

and we verify the formulas of Ricci type

$$u_{b_1\ |a|b} - u_{b_1\ |b|a} + (\rho, h)\, \mathbb{T}^{d}{}_{ab} u_{b_1\ |d} = -u_d\, (\rho, h)\, \mathbb{R}_{b_1\ ab}^{d} - u_{b_1\ |d} L_{ab}^{d} \circ h \tag{5.17}$$

In particular, if $h = Id_M$*, then the relations* (5.17) *become*

$$u_{b_1\ |a|b} - u_{b_1\ |b|a} + \rho\mathbb{T}^{d}{}_{ab} u_{b_1\ |d} = -u_d \rho \mathbb{R}_{b_1\ ab}^{d} - u_{b_1\ |d} L_{ab}^{d}. \tag{5.17'}$$

Moreover, if $\rho = id_{TM}$ *then the relations* (5.17′) *become*

$$u_{j_1\ |i|j} - u_{j_1\ |j|i} + \mathbb{T}^{h}{}_{ij} u_{j_1\ |h} = u_h \mathbb{R}_{j_1\ ij}^{h}. \tag{5.17''}$$

Theorem 5.5 *For any tensor field*

$$T_{b_1,\ldots,b_q}^{a_1,\ldots,a_p} s_{a_1} \otimes \ldots \otimes s_{a_p} \otimes s^{b_1} \otimes \ldots \otimes s^{b_q}\ ,$$

we verify the equality:

$$
\begin{array}{ll}
(5.18) & \begin{array}{l}
T_{b_1,\ldots,b_q|\alpha\beta}^{a_1,\ldots,a_p} - T_{b_1,\ldots,b_q|\beta\alpha}^{a_1,\ldots,a_p} = T_{b_1,\ldots,b_q}^{aa_2,\ldots,a_p}\left(\rho,h\right)\mathbb{R}_{a\ \alpha\beta}^{a_1} + \ldots \\
+T_{b_1,\ldots,b_q}^{a_1,\ldots,a_{p-1}a}\left(\rho,h\right)\mathbb{R}_{a\ \alpha\beta}^{a_p} - T_{b,b_2,\ldots,b_q}^{a_1,\ldots,a_p}\left(\rho,h\right)\mathbb{R}_{b_1\ \alpha\beta}^{b} - \ldots \\
-T_{b_1,\ldots,b_{q-1}b}^{a_1,\ldots,a_p}\left(\rho,h\right)\mathbb{R}_{b_q\ \alpha\beta}^{b} - T_{b_1,\ldots,b_q|\gamma}^{a_1,\ldots,a_p}L_{\alpha\beta}^{\gamma}\circ h.
\end{array}
\end{array}
$$

In particular, if $h=Id_M$*, then the relations* (5.18) *become*

$$
\begin{array}{ll}
(5.18') & \begin{array}{l}
T_{b_1,\ldots,b_q|\alpha\beta}^{a_1,\ldots,a_p} - T_{b_1,\ldots,b_q|\beta\alpha}^{a_1,\ldots,a_p} = T_{b_1,\ldots,b_q}^{aa_2,\ldots,a_p}\rho\mathbb{R}_{a\ \alpha\beta}^{a_1} + \ldots \\
+T_{b_1,\ldots,b_q}^{a_1,\ldots,a_{p-1}a}\rho\mathbb{R}_{a\ \alpha\beta}^{a_p} - T_{b,b_2,\ldots,b_q}^{a_1,\ldots,a_p}\rho\mathbb{R}_{b_1\ \alpha\beta}^{b} - \ldots \\
-T_{b_1,\ldots,b_{q-1}b}^{a_1,\ldots,a_p}\rho\mathbb{R}_{b_q\ \alpha\beta}^{b} - T_{b_1,\ldots,b_q|\gamma}^{a_1,\ldots,a_p}L_{\alpha\beta}^{\gamma}.
\end{array}
\end{array}
$$

Theorem 5.6 *If* $(E,\pi,M)=(F,\nu,N)$*, then we obtain the following formulas of Ricci type:*

$$
\begin{array}{ll}
(5.19) & \begin{array}{l}
T_{b_1,\ldots,b_q\ |b|c}^{a_1,\ldots,a_p} - T_{b_1,\ldots,b_q|c|b}^{a_1,\ldots,a_p} + \left(\rho,h\right)\mathbb{T}_{\ bc}^{d}T_{b_1,\ldots,b_q|d}^{a_1,\ldots,a_p} \\
= T_{b_1,\ldots,b_q}^{aa_2,\ldots,a_p}\left(\rho,h\right)\mathbb{R}_{a\ bc}^{a_1} + \ldots + T_{b_1,\ldots,b_q}^{a_1,\ldots,a_{p-1}a}\left(\rho,h\right)\mathbb{R}_{a\ bc}^{a_p} \\
-T_{b,b_2,\ldots,b_q}^{a_1,\ldots,a_p}\left(\rho,h\right)\mathbb{R}_{b_1\ bc}^{b} - \ldots - T_{b_1,\ldots,b_{q-1}b}^{a_1,\ldots,a_p}\left(\rho,h\right)\mathbb{R}_{b_q\ bc}^{b} - T_{b_1,\ldots,b_q|d}^{a_1,\ldots,a_p}L_{bc}^{d}\circ h.
\end{array}
\end{array}
$$

In particular, if $h=Id_M$*, then the relations* (5.21) *become*

$$
\begin{array}{ll}
(5.19') & \begin{array}{l}
T_{b_1,\ldots,b_q\ |b|c}^{a_1,\ldots,a_p} - T_{b_1,\ldots,b_q|c|b}^{a_1,\ldots,a_p} + \rho\mathbb{T}_{\ bc}^{d}T_{b_1,\ldots,b_q|d}^{a_1,\ldots,a_p} \\
= T_{b_1,\ldots,b_q}^{aa_2,\ldots,a_p}\rho\mathbb{R}_{a\ bc}^{a_1} + \ldots + T_{b_1,\ldots,b_q}^{a_1,\ldots,a_{p-1}a}\rho\mathbb{R}_{a\ bc}^{a_p} \\
-T_{b,b_2,\ldots,b_q}^{a_1,\ldots,a_p}\rho\mathbb{R}_{b_1\ bc}^{b} - \ldots - T_{b_1,\ldots,b_{q-1}b}^{a_1,\ldots,a_p}\rho\mathbb{R}_{b_q\ bc}^{b} - T_{b_1,\ldots,b_q|d}^{a_1,\ldots,a_p}L_{bc}^{d}.
\end{array}
\end{array}
$$

We observe that if the structure functions of generalized Lie algebroid

$$\Big(\left(F,\nu,M\right),\left[,\right]_{F,h},\left(\rho,Id_M\right)\Big),$$

the (ρ,h)-torsion associated to linear ρ-connection $\rho\Gamma$ and the (ρ,h)-curvature associated to linear ρ-connection $\rho\Gamma$ are null, then we have the equality:

$$
\begin{array}{ll}
(5.20) & T_{b_1,\ldots,b_q|b|c}^{a_1,\ldots,a_p} = T_{b_1,\ldots,b_q|c|b}^{a_1,\ldots,a_p},
\end{array}
$$

which generalizes the Schwartz equality.

Theorem 5.7 *If* (E,π,M)=(F,ν,N)*, then the following relations hold good*

$$
\begin{array}{ll}
(\tilde{B}_1) & \displaystyle\sum_{cyclic(u_1,u_2,u_3)}\Big\{\rho\ddot{D}_{U_1}\left(\left(\rho,h\right)\mathbb{T}\left(U_2,U_3\right)\right) - \left(\rho,h\right)\mathbb{R}\left(U_1,U_2\right)U_3 \\
& \qquad + \left(\rho,h\right)\mathbb{T}\left(\left(\rho,h\right)\ \mathbb{T}\left(U_1,U_2\right),U_3\right)\} = 0,
\end{array}
$$

and

$$
\begin{array}{ll}
(\tilde{B}_2) & \displaystyle\sum_{cyclic(u_1,u_2,u_3,u)}\Big\{\rho\ddot{D}_{U_1}\left(\left(\rho,h\right)\mathbb{R}\left(U_2,U_3\right)U\right) - \left(\rho,h\right)\mathbb{R}\left(\left(\rho,h\right)\mathbb{T}\left(U_1,U_2\right),U_3\right)U\Big\} = 0.
\end{array}
$$

respectively. This identities will be called the first respectively the second identity of Bianchi type.

In particular, if $h = Id_M$*, then the identities* $(\tilde{B}_1)$ *and* $(\tilde{B}_2)$ *become*

$$(\tilde{B}_1') \qquad \sum_{cyclic(u_1,u_2,u_3)} \{\rho D_{u_1}(\rho \mathbb{T}(u_2,u_3)) - \rho \mathbb{R}(u_1,u_2)u_3 + \rho \mathbb{T}(\rho\ \mathbb{T}(u_1,u_2),u_3)\} = 0,$$

$$(\tilde{B}_2') \qquad \sum_{cyclic(u_1,u_2,u_3,u)} \{\rho D_{u_1}(\rho \mathbb{R}(u_2,u_3)u) - \rho \mathbb{R}(\rho \mathbb{T}(u_1,u_2),u_3)u\} = 0.$$

which will be called the first respectively the second identity of Bianchi type.

Remark 5.2 On components, the identities of Bianchi type $(\tilde{B}_1)$ and $(\tilde{B}_2)$ become:

$$(\tilde{B}_1'') \qquad \begin{array}{l} \sum\limits_{cyclic(a_1,a_2,a_3)} \left\{ (\rho,h)\,\mathbb{T}^a{}_{a_2a_3|a_1} + (\rho,h)\,\mathbb{T}^a{}_{ga_3} \cdot (\rho,h)\,\mathbb{T}^g{}_{a_1a_2} \right\} \\ \qquad\qquad = \sum\limits_{cyclic(a_1,a_2,a_3)} (\rho,h)\,\mathbb{R}^a_{a_3\ a_1a_2} \end{array}$$

and

$$(\tilde{B}_2'') \qquad \sum_{cyclic(a_1,a_2,a_3)} \left\{ (\rho,h)\,\mathbb{R}^a_{b\ a_2a_3|a_1} + (\rho,h)\,\mathbb{R}^a_{b\ ga_3} \cdot (\rho,h)\,\mathbb{T}^g{}_{a_1a_2} \right\} = 0.$$

If the (ρ,h)-torsion is null, then the identities of Bianchi type become:

$$(\tilde{B}_1''') \qquad \sum_{cyclic(a_1\ a_2,a_3)} (\rho,h)\,\mathbb{R}^a_{a_3\ a_1a_2} = 0$$

and

$$(\tilde{B}_2''') \qquad \sum_{cyclic(a_1,a_2,a_3)} (\rho,h)\,\mathbb{R}^a_{b\ a_2a_3|a_1} = 0.$$

6 (Pseudo)metrizable vector bundles. Formulas of Levi-Civita type

We will apply our theory for the diagram:

$$(6.1) \qquad \begin{array}{ccc} E & & \left(F,[,]_{F,h},(\rho,Id_N)\right) \\ \pi\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & N \end{array},$$

where $(E,\pi,M) \in |\mathbf{B^v}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,Id_N)\right) \in |\mathbf{GLA}|$.

Definition 6.1 We will say that the *vector bundle* (E,π,M) *is endowed with a pseudometrical structure* if it exists $g = g_{ab}s^a \otimes s^b \in \mathcal{T}_2^0(E,\pi,M)$ such that for each $x \in M$, the matrix $\|g_{ab}(x)\|$ is nondegenerate and symmetric.

Moreover, if for each $x \in M$ the matrix $\|g_{ab}(x)\|$ has constant signature, then we will say that *the vector bundle* (E,π,M) *is endowed with a metrical structure.*

If $g = g_{ab}s^a \otimes s^b \in \mathcal{T}_2^0(E,\pi,M)$ is a (pseudo) metrical structure, then, for any $a,b \in \overline{1,r}$ and for any vector local $(m+r)$-chart (U,s_U) of (E,π,M), we consider the real functions

$$U \xrightarrow{\tilde{g}^{ba}} \mathbb{R}$$

such that $\left\|\tilde{g}^{ba}(x)\right\| = \left\|g_{ab}(x)\right\|^{-1}$, for any $\forall x \in U$.

Definition 6.2 We admit that (E,π,M) is a vector bundle endowed with a (pseudo)metrical structure g and with a linear ρ-connection $\rho\Gamma$.

We will say that the *linear ρ-connection $\rho\Gamma$ is compatible with the (pseudo)metrical structure g* if

$$\rho D_z g = 0,\ \forall z \in \Gamma(F,\nu,N). \tag{6.2}$$

Definition 6.3 We will say that the vector bundle (E,π,M) is *ρ-(pseudo)metrizable,* if it exists a (pseudo)metrical structure $g \in \mathcal{T}_2^0(E,\pi,M)$ and a linear ρ-connection $\rho\Gamma$ for (E,π,M) compatible with g. The id_{TM}-(pseudo)metrizable vector bundles will be called *(pseudo)metrizable vector bundles.*

In particular, if (TM,τ_M,M) is a (pseudo)metrizable vector bundle, then we will say that (TM,τ_M,M) *is a* (pseudo)Riemannian space*,* and the manifold M will be called *(pseudo)Riemannian manifold.*

The linear connection of a (pseudo)Riemannian space will be called *(pseudo)Riemannian linear connection.*

Theorem 6.1 *If $(E,\pi,M) = (F,\nu,N)$ and $g \in \mathcal{T}_2^0(h^*E,h^*\pi,M)$ is a (pseudo)metrical structure, then the local real functions*

$$\begin{aligned}\rho\Gamma_{bc}^a &= \frac{1}{2}\tilde{g}^{ad}\left(\rho_c^k \circ h\frac{\partial g_{bd}}{\partial x^k} + \rho_b^j \circ h\frac{\partial g_{dc}}{\partial x^j} - \rho_d^h \circ h\frac{\partial g_{bc}}{\partial x^h}\right.\\ &\left.+ g_{ec}L_{bd}^e \circ h + g_{be}L_{dc}^e \circ h - g_{de}L_{bc}^e \circ h\right).\end{aligned} \tag{6.3}$$

are the components of a linear ρ-connection $\rho\Gamma$ for the vector bundle $(h^*E,h^*\pi,M)$ compatible with g such that $(\rho,h)\mathbb{T} = 0$.

Therefore, the vector bundle $(h^*E,h^*\pi,M)$ becomes ρ-(pseudo)metrizable. The linear ρ-connection $\rho\Gamma$ will be called *linear ρ-connection of Levi-Civita type.*

Proof. Since

$$\begin{aligned}\left(\rho\ddot{D}_U g\right)V \otimes Z &= \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)\left((g(V\otimes Z)) - g\left(\left(\rho\ddot{D}_U V\right)\otimes Z\right)\right.\\ &\left.-g\left(V \otimes \left(\rho\ddot{D}_U Z\right)\right)\right),\ \forall U,V,Z \in \Gamma(h^*E,h^*\pi,M).\end{aligned}$$

it results that, for any $U,V,Z \in \Gamma(h^*E,h^*\pi,M)$, we obtain the equalities:

$$\begin{array}{ll}(1) & \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)(g(V\otimes Z)) = g\left(\left(\rho\ddot{D}_U V\right)\otimes Z\right) + g\left(V\otimes\left(\rho\ddot{D}_U Z\right)\right),\\ (2) & \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(Z)(g(U\otimes V)) = g\left(\left(\rho\ddot{D}_Z U\right)\otimes V\right) + g\left(U\otimes\left(\rho\ddot{D}_Z V\right)\right),\\ (3) & \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(V)(g(Z\otimes U)) = g\left(\left(\rho\ddot{D}_V Z\right)\otimes U\right) + g\left(Z\otimes\left(\rho\ddot{D}_V U\right)\right).\end{array}$$

We observe that $(1)+(3)-(2)$ is equivalent with the equality:

$$\begin{array}{l} g\left(\left(\rho\ddot{D}_{U}V+\rho\ddot{D}_{V}U\right)\otimes Z\right)+g\left(\left(\rho\ddot{D}_{V}Z-\rho\ddot{D}_{Z}V\right)\otimes U\right)\\ \quad +g\left(\left(\rho\ddot{D}_{U}Z-\rho\ddot{D}_{Z}U\right)\otimes V\right)=\Gamma\left(\overset{h^{*}E}{\rho},Id_{M}\right)(U)\left(g\left(V\otimes Z\right)\right)\\ \quad +\Gamma\left(\overset{h^{*}E}{\rho},Id_{M}\right)(V)\left(g\left(Z\otimes U\right)\right)-\ \Gamma\left(\overset{h^{*}E}{\rho},Id_{M}\right)(Z)\left(g\left(U\otimes V\right)\right). \end{array}$$

Using the condition $(\rho,h)\,\mathbb{T}=0$, which is equivalent with the equality

$$\rho\ddot{D}_{U}V-\rho\ddot{D}_{V}U-[U,V]_{h^{*}E}=0,$$

we obtain:

$$\begin{array}{ll} 2g\left(\left(\rho\ddot{D}_{U}V\right)\otimes Z\right) & =\Gamma\left(\overset{h^{*}E}{\rho},Id_{M}\right)(U)\cdot\left(g\left(V\otimes Z\right)\right)\\ & +\Gamma\left(\overset{h^{*}E}{\rho},Id_{M}\right)(V)\left(g\left(Z\otimes U\right)\right)-\Gamma\left(\overset{h^{*}E}{\rho},Id_{M}\right)(Z)\left(g\left(U\otimes V\right)\right)\\ & +g\left([U,V]_{h^{*}E}\otimes Z\right)-g\left([U,Z]_{h^{*}E}\otimes V\right)\\ & -g\left([V,Z]_{h^{*}E}\otimes U\right),\ \forall U,V,Z\in\Gamma\left(h^{*}E,h^{*}\pi,M\right). \end{array}$$

Therefore, we obtain the equality:

$$\begin{array}{l} 2g\left(\left(\rho\Gamma_{ba}^{d}S_{d}\right)\otimes S_{c}\right)=\rho_{a}^{i}\circ h\dfrac{\partial g\left(S_{b}\otimes S_{c}\right)}{\partial x^{i}}+\ \rho_{b}^{j}\circ h\dfrac{\partial g\left(S_{c}\otimes S_{a}\right)}{\partial x^{j}}-\rho_{c}^{k}\circ h\dfrac{\partial g\left(S_{a}\otimes S_{b}\right)}{\partial x^{k}}\\ \quad +g\left(\left(L_{ab}^{d}\circ h\right)S_{d}\otimes S_{c}\right)-g\left(\left(L_{ac}^{d}\circ h\right)S_{d}\otimes S_{b}\right)-g\left(\left(L_{bc}^{d}\circ h\right)S_{d}\otimes S_{a}\right), \end{array}$$

which is equivalent with:

$$\begin{array}{l} 2g_{dc}\rho\Gamma_{ba}^{d}=\rho_{a}^{i}\circ h\dfrac{\partial g_{bc}}{\partial x^{i}}+\rho_{b}^{j}\circ h\dfrac{\partial g_{ca}}{\partial x^{j}}-\rho_{c}^{k}\circ h\dfrac{\partial g_{ab}}{\partial x^{k}}+\left(L_{ab}^{d}\circ h\right)g_{dc}\\ \quad -\left(L_{ac}^{d}\circ h\right)g_{db}-\left(L_{bc}^{d}\circ h\right)g_{da}. \end{array}$$

Finally, we obtain:

$$\begin{array}{ll} \rho\Gamma_{ba}^{d} & =\dfrac{1}{2}\tilde{g}^{dc}\left(\rho_{a}^{i}\circ h\dfrac{\partial g_{bc}}{\partial x^{i}}+\rho_{b}^{j}\circ h\dfrac{\partial g_{ca}}{\partial x^{j}}-\rho_{c}^{k}\circ h\dfrac{\partial g_{ab}}{\partial x^{k}}\right.\\ & \left.+\left(L_{ab}^{d}\circ h\right)g_{dc}-\left(L_{ac}^{d}\circ h\right)g_{db}-\left(L_{bc}^{d}\circ h\right)g_{da}\right), \end{array}$$

where $\left\|g^{dc}(x)\right\|=\left\|g_{cd}(x)\right\|^{-1}$, for any $x\in M$. *q.e.d.*

Corollary 6.1 *In particular, if* $h=Id_{M}$, $(E,\pi,M)=(F,\nu,N)$ *and* $g\in\mathcal{T}_{2}^{0}(E,\pi,M)$ *is a (pseudo)metrical structure, then the local real functions*

$$(6.3')\qquad \rho\Gamma_{bc}^{a}=\tfrac{1}{2}\tilde{g}^{ad}\left(\rho_{c}^{k}\tfrac{\partial g_{bd}}{\partial x^{k}}+\rho_{b}^{j}\tfrac{\partial g_{dc}}{\partial x^{j}}-\rho_{d}^{h}\tfrac{\partial g_{bc}}{\partial x^{h}}+g_{ec}L_{bd}^{e}+g_{be}L_{dc}^{e}-g_{de}L_{bc}^{e}\right).$$

are the components of a linear ρ-connection $\rho\Gamma$ for the vector bundle (E,π,M) compatible with g such that $\rho\mathbb{T}=0$.

Therefore, the vector bundle (E,π,M) becomes ρ-(pseudo)metrizable.

The linear ρ-connection $\rho\Gamma$ will be called *linear ρ-connection of Levi-Civita type.*

In particular, if $\rho=Id_{TM}$, we obtain the classical Levi-Civita connection.

Theorem 6.2. *If $(E,\pi,M)=(F,\nu,N)$, $g\in\mathcal{T}_2^0\,(h^*E,h^*\pi,M)$ is a pseudo(metrical) structure and $\mathbb{T}\in\mathcal{T}_2^1\,(h^*E,h^*\pi,M)$ such that its components are skew symmetric in the lover indices, then the local real functions*

$$\rho\overset{\circ}{\Gamma}{}^a_{bc}=\rho\Gamma^a_{bc}+\frac{1}{2}\tilde{g}^{ad}\left(g_{de}\mathbb{T}^e_{bc}-g_{be}\mathbb{T}^e_{dc}+g_{ec}\mathbb{T}^e_{bd}\right), \tag{6.4}$$

are the components of a linear ρ-connection compatible with the (pseudo) metrical structure g, where $\rho\Gamma^a_{bc}$ are the components of linear ρ-connection of Levi-Civita type (6.3)*. Therefore, the vector bundle $(h^*E,h^*\pi,M)$ becomes ρ-(pseudo)metrizable. In addition, the tensor field $\mathbb{T}$ is the (ρ,h)-torsion tensor field.*

Corollary 6.2 *In particular, if $h=Id_M$, $(E,\pi,M)=(F,\nu,M)$, $g\in\mathcal{T}_2^0\,(E,\pi,M)$ is a pseudo(metrical) structure and $T\in\mathcal{T}_2^1\,(E,\pi,M)$ such that its components are skew symmetric in the lover indices, then the local real functions*

$$\rho\overset{\circ}{\Gamma}{}^a_{bc}=\rho\Gamma^a_{bc}+\frac{1}{2}\tilde{g}^{ad}\left(g_{de}\mathbb{T}^e_{bc}-g_{be}\mathbb{T}^e_{dc}+g_{ec}\mathbb{T}^e_{bd}\right), \tag{6.4$'$}$$

are the components of a linear ρ-connection compatible with the (pseudo)metrical structure g, where $\rho\Gamma^a_{bc}$ are the components of linear ρ-connection of Levi-Civita type (6.3$'$)*. Therefore, the vector bundle (E,π,M) becomes ρ-(pseudo)metrizable. In addition, the tensor field $\mathbb{T}$ is the ρ-torsion tensor field.*

Theorem 6.3 *If $g\in\mathcal{T}_2^0\,(E,\pi,M)$ is a pseudo (metrical) structure and $\rho\overset{\circ}{\Gamma}$ is the linear ρ-connection* (6.4) *for the vector bundle (E,π,M), then the local real functions*

$$\rho\overset{k}{\Gamma}{}^{a}_{b\alpha}=\rho\overset{\circ}{\Gamma}{}^a_{b\alpha}+\tfrac{1}{2}\tilde{g}^{ac}g_{\underset{\circ}{cb|}\alpha} \tag{6.5}$$

are the components of a linear ρ-connection compatible with the (pseudo) metrical structure g. Therefore, the vector bundle (E,π,M) becomes ρ-(pseudo)metrizable.

Theorem 6.4 *If $g\in\mathcal{T}_2^0\,(E,\pi,M)$ is a pseudo (metrical) structure, $\rho\overset{\circ}{\Gamma}$ is the linear ρ-connection* (6.4) *for the vector bundle (E,π,M) and $T=T^d_{c\alpha}s_d\otimes s^c\otimes t^\alpha$, then the local real functions*

$$\rho\Gamma^a_{b\alpha}=\rho\overset{k}{\Gamma}{}^{a}_{b\alpha}+\tfrac{1}{2}O^{ca}_{bd}T^d_{c\alpha}, \tag{6.6}$$

are the components of a linear ρ-connection compatible with (pseudo) metrical structure g, where $O^{ca}_{bd}=\frac{1}{2}\delta^c_b\delta^a_d-g_{bd}\tilde{g}^{ca}$ is the Obata operator. Therefore, the vector bundle (E,π,M) becomes ρ-(pseudo)metrizable.

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ALGEBRAIC CONSTRUCTIONS IN THE CATEGORY OF LIE ALGEBROIDS

by
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Abstract

A generalized notion of a Lie algebroid is presented. Using this, the Lie algebroid generalized tangent bundle is obtained. A new point of view over (linear) connections theory on a fiber bundle is presented. These connections are characterized by o horizontal distribution of the Lie algebroid generalized tangent bundle. Some basic properties of these generalized connections are investigated. Special attention to the class of linear connections is paid. The recently studied Lie algebroids connections can be recovered as special cases within this more general framework. In particular, all results are similar with the classical results. Formulas of Ricci and Bianchi type and linear connections of Levi-Civita type are presented.

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Contents

1 Introduction

The theory of connections constitutes one of the most important chapter of differential geometry, which has been explored in the literature (see [2, 3, 4, 5, 10, 11, 12, 13, 14, 15, 16]). Connections theory has become an indispensable tool in various branches of theoretical and mathematical physics.

If (E, π, M) is a fiber bundle with paracompact base and (VTE, τ_E, E) is the kernel vector bundle of the tangent $\mathbf{B^v}$-morphism $(T\pi, \pi)$, then we obtain the short exact sequence

$$(1)\qquad \begin{array}{ccccccccc} 0 & \hookrightarrow & VTE & \hookrightarrow & TE & \xrightarrow{\pi!} & \pi^*TM & \longrightarrow & 0 \\ & & \downarrow \tau_E & & \downarrow \tau_E & & \downarrow \pi^*\tau_M & & \\ & & E & \xrightarrow{Id_E} & E & \xrightarrow{Id_E} & E & & \end{array}$$

where $\pi!$ is the projection of TE onto π^*TM.

We know that a split to the right in the previous short exact sequence, i.e. a smooth map $h \in \mathbf{Man}(\pi^*TM, TE)$ so that $\pi! \circ h = Id_{\pi^*TM}$, is a *connection in the Ehresmann sense.*

If (HTE, τ_E, E) is the image vector bundle of the $\mathbf{B^v}$-morphism (h, Id_E), then the tangent vector bundle (TE, τ_E, E) is a Whitney sum between the *horizontal vector bundle* (HTE, τ_E, E) and the *vertical vector bundle* (VTE, τ_E, E).

From the above notion of connection, one can easily derive more specific types of connections by imposing additional conditions. In the literature one can find several generalizations of the concept of Ehresmann connection obtained by relaxing the conditions on the horizontal vector bundle.

- First of all, we are thinking here of the so-called *partial connections,* where (HTE, τ_E, E) does not determine a full complement of (VTE, τ_E, E). More precisely, $\Gamma(HTE, \tau_E, E)$ has zero intersection with $\Gamma(VTE, \tau_E, E)$, but (HTE, τ_E, E) projects onto a vector subbundle of (TM, τ_M, M). (see [7])

- Secondly, there also exists a notion of *pseudo-connection*, introduced under the name of *quasi-connection* in a paper by Y. C. Wong [16]. Linear pseudo-connections and generalization of it have been studied by many authors. (see [3])

P. Popescu build the *relativ tangent space* and using that he obtained a new *generalized connection* on a vector bundle.[11] (see also [12])

In the paper [4] by R. L. Fernandez a *contravariant connection* in the framework of Poisson Geometry there are presented. Given a Poisson manifold M with tensor Λ which does not have to be of constant rank, a *covariant connection* on the principal bundle (P, π, M) is a G-invariant bundle map $h \in \mathbf{Man}(\pi^*(T^*M), TP)$ so that the diagram is commutative

$$(2)\qquad \begin{array}{ccc} \pi^*(T^*M) & \xrightarrow{h} & TP \\ \pi^*\left(\overset{*}{\tau}_M\right)\downarrow & & \downarrow T\pi \\ T^*M & \xrightarrow{\sharp_\Lambda} & TM \end{array}$$

where $(\sharp_\Lambda, Id_M)$ is the natural vector bundle morphism induced by the Poisson tensor. In the paper [5], R. L. Fernandez has extending this theory by replacing the cotangent

bundle of a Poisson manifold by a Lie algebroid over an arbitrary manifold and the map $\sharp_\Lambda$ by the anchor map of the Lie algebroid. This resulted into a notion of *Lie algebroid connection* which, in particular, turns out to be appropriate for studying the geometry of singular distributions.

B. Langerock and F. Cantrijn [2] proposed a *general notion of connection* on a fiber bundle (E,π,M), as being a smooth linear bundle map $h\in\mathbf{Man}\left(\pi^*(F),TE\right)$ so that the diagram is commutative

$$\begin{array}{ccc}\pi^*(F) & \xrightarrow{h} & TE\\ \downarrow & & \downarrow T\pi\\ F & \xrightarrow{\rho} & TM\end{array}\tag{3}$$

where (F,ν,M) is an arbitrary vector bundle and (ρ, Id_M) is a vector bundle morphism of (F,ν,M) source and (TM,τ_M,M) target.

Different equivalent definitions of a (linear) connection on a vector bundle are known and there are in current usage. We know the following

Theorem *If we have a short exact sequence of vector bundles over a paracompact manifold M*

$$\begin{array}{ccccccc}0\hookrightarrow & E' & \xrightarrow{f} & E & \xrightarrow{g} & E'' & \longrightarrow 0\\ & \downarrow\pi' & & \downarrow\pi & & \downarrow\pi'' & \\ & M & \xrightarrow{Id_M} & M & \xrightarrow{Id_M} & M & \end{array}\tag{4}$$

then there exists a right split if and only if there exists a left split.

So, a split to the left in the short exact sequence (1), i.e. a smooth map $\Gamma\in\mathbf{Man}\,(TE,VTE)$ so that $\Gamma\circ i=Id_{TE}$, is an equivalent definition with the Ehresmann connection.

We remark that the secret of the Ehresmann connection is given by the diagrams

$$\begin{array}{cccccc}E & & (TM,[,]_{TM}) & \xrightarrow{Id_{TM}} & (TM,[,]_{TM})\\ \downarrow\pi & & \downarrow\tau_M & & \downarrow\tau_M\\ M & \xrightarrow{Id_M} & M & \xrightarrow{Id_M} & M\end{array}\tag{5}$$

where (E,π,M) is a fiber bundle and $\left((TM,\tau_M,M),[,]_{TM},(Id_{TM},Id_M)\right)$ is the standard Lie algebroid.

First time, appeared the idea to change the standard Lie algebroid with an arbitrary Lie algebroid as in the diagrams

$$\begin{array}{cccccc}E & & (F,[,]_F) & \xrightarrow{\rho} & (TM,[,]_{TM})\\ \downarrow\pi & & \downarrow\nu & & \downarrow\tau_M\\ M & \xrightarrow{Id_M} & M & \xrightarrow{Id_M} & M\end{array}\tag{6}$$

Second time, appeared the idea to change in the previous diagrams the identities morphisms with arbitrary **Man**-isomorphisms h and η as in the diagrams

$$\begin{array}{cccccccc}E & & \left(F,[,]_{F,h}\right) & \xrightarrow{\rho} & (TM,[,]_{TM}) & \xrightarrow{Th} & (TN,[,]_{TN})\\ \downarrow\pi & & \downarrow\nu & & \downarrow\tau_M & & \downarrow\tau_N\\ M & \xrightarrow{h} & N & \xrightarrow{\eta} & M & \xrightarrow{h} & N\end{array}\tag{7}$$

where
$$(\rho, \eta) \in \mathbf{B^v}\left(\left(F, \nu, M\right), \left(TM, \tau_M, M\right)\right)$$
and
$$\begin{array}{ccc}\Gamma\left(F, \nu, N\right) \times \Gamma\left(F, \nu, N\right) & \xrightarrow{[,]_{F,h}} & \Gamma\left(F, \nu, N\right) \\ (u, v) & \longmapsto & [u, v]_{F,h}\end{array}$$
is an operation with the following properties:

GLA_1. *the equality holds good*
$$[u, f \cdot v]_{F,h} = f\,[u, v]_{F,h} + \Gamma\left(Th \circ \rho, h \circ \eta\right)(u)\, f \cdot v,$$
for all $u, v \in \Gamma\left(F, \nu, N\right)$ *and* $f \in \mathcal{F}\left(N\right).$

GLA_2. *the* 4*-tuple* $\left(\Gamma\left(F, \nu, N\right), +, \cdot, [,]_{F,h}\right)$ *is a Lie* $\mathcal{F}\left(N\right)$*-algebra,*

GLA_3. *the* **Mod***-morphism* $\Gamma\left(Th \circ \rho, h \circ \eta\right)$ *is a* **LieAlg***-morphism of*
$$\left(\Gamma\left(F, \nu, N\right), +, \cdot, [,]_{F,h}\right)$$
source and
$$\left(\Gamma\left(TN, \tau_N, N\right), +, \cdot, [,]_{TN}\right)$$
target.

So, appeared the notion of *generalized Lie algebroid* which is presented in *Definition 2.1.* Using this new notion we build the *Lie algebroid generalized tangent bundle* in the *Theorem 3.1.* Particularly, if $\left(\left(F, \nu, N\right), [,]_F, \left(\rho, Id_N\right)\right)$ is a Lie algebroid, $\left(E, \pi, M\right) = \left(F, \nu, N\right)$ and $h = Id_M$, then we obtain a similar Lie algebroid with the the *prolongation Lie algebroid.* (see [6, 8, 9, 10]) Using this general framework, in Section 4, we propose and develop a (linear) connections theory of Ehresmann type for fiber bundles in general and for vector bundles in particular. It covers all types of connections mentioned. In this general framework, we can define the covariant derivatives of sections of a fiber bundle $\left(E, \pi, M\right)$ with respect to sections of a generalized Lie algebroid
$$\left(\left(F, \nu, N\right), [,]_{F,h}, \left(\rho, \eta\right)\right).$$

In particular, if we use the generalized Lie algebroid structure
$$\left([,]_{TM, Id_M}, \left(Id_{TM}, Id_M\right)\right)$$
for the tangent bundle $\left(TM, \tau_M, M\right)$ in our theory, then the linear connections obtained are similar with the classical linear connections.

It is known that in Yang-Mills theory the set
$$Cov^0_{(E,\pi,M)}$$
of covariant derivatives for the vector bundle $\left(E, \pi, M\right)$ such that
$$X\left(\langle u, v\rangle_E\right) = \langle D_X\left(u\right), v\rangle_E + \langle u, D_X\left(v\right)\rangle_E,$$
for any $X \in \mathcal{X}\left(M\right)$ and $u, v \in \Gamma\left(E, \pi, M\right)$, is very important, because the Yang-Mills theory is a variational theory which use (see [1]) the Yang-Mills functional
$$\begin{array}{rcl}Cov^0_{(E,\pi,M)} & \xrightarrow{\mathcal{YM}} & \mathbb{R} \\ D_X & \longmapsto & \dfrac{1}{2}\int\limits_M \left\|\mathbb{R}^{D_X}\right\|^2 v_g\end{array}$$

where $\mathbb{R}^{D_X}$ is the curvature.

Using our linear connections theory, we succeed to extend the set $Cov^0_{(E,\pi,M)}$ of Yang-Mills theory, because using all generalized Lie algebroid structures for the tangent bundle (TM, τ_M, M), we obtain all possible linear connections for the vector bundle (E, π, M).

More importantly, it may bring within the reach of connection theory certain geometric structures which have not yet been considered from such a point of view. Finally, using our theory of linear connections, the formulas of Ricci and Bianchi type and linear connections of Levi-Civita type are presented.

2 Preliminaries

In general, if $\mathcal{C}$ is a category, then we denote $|\mathcal{C}|$ the class of objects and for any $A, B \in |\mathcal{C}|$, we denote $\mathcal{C}(A, B)$ the set of morphisms of A source and B target. Let **Vect**, **Liealg**, **Mod, Man**, **B** and $\mathbf{B^v}$ be the category of real vector spaces, Lie algebras, modules, manifolds, fiber bundles and vector bundles respectively.

We know that if $(E, \pi, M) \in |\mathbf{B^v}|$, $\Gamma(E, \pi, M) = \{u \in \mathbf{Man}(M, E) : u \circ \pi = Id_M\}$ and $\mathcal{F}(M) = \mathbf{Man}(M, \mathbb{R})$, then $(\Gamma(E, \pi, M), +, \cdot)$ is a $\mathcal{F}(M)$-module. If $(\varphi, \varphi_0) \in \mathbf{B^v}((E, \pi, M), (E', \pi', M'))$ such that $\varphi_0 \in Iso_{\mathbf{Man}}(M, M')$, then, using the operation

$$\begin{array}{ccc} \mathcal{F}(M) \times \Gamma(E', \pi', M') & \xrightarrow{\cdot} & \Gamma(E', \pi', M') \\ (f, u') & \longmapsto & f \circ \varphi_0^{-1} \cdot u' \end{array}$$

it results that $(\Gamma(E', \pi', M'), +, \cdot)$ is a $\mathcal{F}(M)$-module and we obtain the **Mod**-morphism

$$\begin{array}{ccc} \Gamma(E, \pi, M) & \xrightarrow{\Gamma(\varphi, \varphi_0)} & \Gamma(E', \pi', M') \\ u & \longmapsto & \Gamma(\varphi, \varphi_0) u \end{array}$$

defined by

$$\Gamma(\varphi, \varphi_0) u(y) = \varphi\left(u_{\varphi_0^{-1}(y)}\right),$$

for any $y \in M'$.

We know that a Lie algebroid is a vector bundle $(F, \nu, N) \in |\mathbf{B^v}|$ such that there exists

$$(\rho, Id_N) \in \mathbf{B^v}((F, \nu, N), (TN, \tau_N, N))$$

and an operation

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[,]_F} & \Gamma(F, \nu, N) \\ (u, v) & \longmapsto & [u, v]_F \end{array}$$

with the following properties:

LA_1. the equality holds good

$$[u, f \cdot v]_F = f[u, v]_F + \Gamma(\rho, Id_N)(u) f \cdot v,$$

for all $u, v \in \Gamma(F, \nu, N)$ and $f \in \mathcal{F}(N)$,

LA_2. the 4-tuple $(\Gamma(F, \nu, N), +, \cdot, [,]_F)$ is a Lie $\mathcal{F}(N)$-algebra,

LA_3. the **Mod**-morphism $\Gamma(\rho, Id_N)$ is a **LieAlg**-morphism of

$$\left(\Gamma(F,\nu,N),+,\cdot,[,]_F\right)$$

source and

$$\left(\Gamma(TN,\tau_N,N),+,\cdot,[,]_{TN}\right)$$

target.

Definition 2.1 Let $M,N\in|\mathbf{Man}|$, $h\in Iso_{\mathbf{Man}}(M,N)$ and $\eta\in Iso_{\mathbf{Man}}(N,M)$. If $(F,\nu,N)\in|\mathbf{B^v}|$ so that there exists

$$(\rho,\eta)\in\mathbf{B^v}\left((F,\nu,N),(TM,\tau_M,M)\right)$$

and an operation

$$\begin{array}{ccc}\Gamma(F,\nu,N)\times\Gamma(F,\nu,N) & \xrightarrow{[,]_{F,h}} & \Gamma(F,\nu,N)\\ (u,v) & \longmapsto & [u,v]_{F,h}\end{array}$$

with the following properties:

GLA_1. the equality holds good

$$[u,f\cdot v]_{F,h}=f[u,v]_{F,h}+\Gamma(Th\circ\rho,h\circ\eta)(u)f\cdot v,$$

for all $u,v\in\Gamma(F,\nu,N)$ and $f\in\mathcal{F}(N)$.

GLA_2. the 4-tuple $\left(\Gamma(F,\nu,N),+,\cdot,[,]_{F,h}\right)$ is a Lie $\mathcal{F}(N)$-algebra,

GLA_3. the **Mod**-morphism $\Gamma(Th\circ\rho,h\circ\eta)$ is a **LieAlg**-morphism of

$$\left(\Gamma(F,\nu,N),+,\cdot,[,]_{F,h}\right)$$

source and

$$\left(\Gamma(TN,\tau_N,N),+,\cdot,[,]_{TN}\right)$$

target, then we will say that *the triple*

$$\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)\tag{2.1}$$

is a generalized Lie algebroid. The couple $\left([,]_{F,h},(\rho,\eta)\right)$ will be called *generalized Lie algebroid structure.*

Remark 2.1 In the particular case, $(\eta,h)=(Id_M,Id_M)$, *we obtain the definition of Lie algebroid.*

Let $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ be a generalized Lie algebroid.

- Locally, for any $\alpha,\beta\in\overline{1,p}$, we set $[t_\alpha,t_\beta]_{F,h}=L_{\alpha\beta}^{\gamma}t_\gamma$. We easily obtain that $L_{\alpha\beta}^{\gamma}=-L_{\beta\alpha}^{\gamma}$, for any $\alpha,\beta,\gamma\in\overline{1,p}$.

The real local functions $L_{\alpha\beta}^{\gamma},\ \alpha,\beta,\gamma\in\overline{1,p}$ will be called the *structure functions of the generalized Lie algebroid* $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$.

- We assume the following diagrams:

$$\begin{array}{ccccc} F & \stackrel{\rho}{\longrightarrow} & TM & \stackrel{Th}{\longrightarrow} & TN \\ \downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\ N & \stackrel{\eta}{\longrightarrow} & M & \stackrel{h}{\longrightarrow} & N \\ & & & & \\ \left(\chi^{\tilde{\imath}}, z^\alpha\right) & & \left(x^i, y^i\right) & & \left(\chi^{\tilde{\imath}}, z^{\tilde{\imath}}\right) \end{array}$$

where $i, \tilde{\imath} \in \overline{1, m}$ and $\alpha \in \overline{1, p}$.

If

$$\left(\chi^{\tilde{\imath}}, z^\alpha\right) \longrightarrow \left(\chi^{\tilde{\imath}\prime}\left(\chi^{\tilde{\imath}}\right), z^{\alpha\prime}\left(\chi^{\tilde{\imath}}, z^\alpha\right)\right),$$
$$\left(x^i, y^i\right) \longrightarrow \left(x^{\acute{\imath}}\left(x^i\right), y^{\acute{\imath}}\left(x^i, y^i\right)\right)$$

and

$$\left(\chi^{\tilde{\imath}}, z^{\tilde{\imath}}\right) \longrightarrow \left(\chi^{\tilde{\imath}\prime}\left(\chi^{\tilde{\imath}}\right), z^{\tilde{\imath}\prime}\left(\chi^{\tilde{\imath}}, z^{\tilde{\imath}}\right)\right),$$

then

$$z^{\acute{\alpha}} = \Lambda_\alpha^{\acute{\alpha}} z^\alpha \ ,$$
$$y^{\acute{\imath}} = \tfrac{\partial x^{\acute{\imath}}}{\partial x^i} y^i$$

and

$$z^{\tilde{\imath}\prime} = \tfrac{\partial \chi^{\tilde{\imath}\prime}}{\partial \chi^{\tilde{\imath}}} z^{\tilde{\imath}}.$$

- We assume that $(\theta, \mu) \stackrel{put}{=} (Th \circ \rho, h \circ \eta)$. If $z^\alpha t_\alpha \in \Gamma(F, \nu, N)$ is arbitrary, then

$$\begin{array}{c} \Gamma\left(Th \circ \rho, h \circ \eta\right)\left(z^\alpha t_\alpha\right) f\left(h \circ \eta\left(\varkappa\right)\right) = \\ = \left(\theta_\alpha^{\tilde{\imath}} z^\alpha \tfrac{\partial f}{\partial \varkappa^{\tilde{\imath}}}\right)\left(h \circ \eta\left(\varkappa\right)\right) = \left(\left(\rho_\alpha^i \circ h\right)\left(z^\alpha \circ h\right) \tfrac{\partial f \circ h}{\partial x^i}\right)\left(\eta\left(\varkappa\right)\right), \end{array} \tag{2.2}$$

for any $f \in \mathcal{F}(N)$ and $\varkappa \in N$.

The coefficients ρ_α^i respectively $\theta_\alpha^{\tilde{\imath}}$ change to $\rho_{\acute{\alpha}}^{\acute{\imath}}$ respectively $\theta_{\acute{\alpha}}^{\tilde{\imath}}$ according to the rule:

$$\rho_{\acute{\alpha}}^{\acute{\imath}} = \Lambda_{\acute{\alpha}}^\alpha \rho_\alpha^i \frac{\partial x^{\acute{\imath}}}{\partial x^i}, \tag{2.3}$$

respectively

$$\theta_{\acute{\alpha}}^{\tilde{\imath}} = \Lambda_{\acute{\alpha}}^\alpha \theta_\alpha^{\tilde{\imath}} \frac{\partial \varkappa^{\tilde{\imath}}}{\partial \varkappa^{\tilde{\imath}}}, \tag{2.4}$$

where

$$\left\|\Lambda_{\acute{\alpha}}^\alpha\right\| = \left\|\Lambda_\alpha^{\acute{\alpha}}\right\|^{-1}.$$

Remark 2.2 The following equalities hold good:

$$\rho_\alpha^i \circ h \frac{\partial f \circ h}{\partial x^i} = \left(\theta_\alpha^{\tilde{\imath}} \frac{\partial f}{\partial \varkappa^{\tilde{\imath}}}\right) \circ h, \forall f \in \mathcal{F}(N). \tag{2.5}$$

and

$$(2.6)\qquad \left(L_{\alpha\beta}^{\gamma}\circ h\right)\left(\rho_{\gamma}^{k}\circ h\right)=\left(\rho_{\alpha}^{i}\circ h\right)\frac{\partial\left(\rho_{\beta}^{k}\circ h\right)}{\partial x^{i}}-\left(\rho_{\beta}^{j}\circ h\right)\frac{\partial\left(\rho_{\alpha}^{k}\circ h\right)}{\partial x^{j}}.$$

Theorem 2.1 *Let $M,N\in|\mathbf{Man}|$, $h\in Iso_{\mathbf{Man}}(M,N)$ and $\eta\in Iso_{\mathbf{Man}}(N,M)$ be. Using the tangent $\mathbf{B^v}$-morphism $(T\eta,\eta)$ and the operation*

$$\begin{array}{ccc}\Gamma(TN,\tau_N,N)\times\Gamma(TN,\tau_N,N) & \xrightarrow{[,]_{TN,h}} & \Gamma(TN,\tau_N,N)\\ (u,v) & \longmapsto & [u,v]_{TN,h}\end{array}$$

where

$$[u,v]_{TN,h}=\Gamma\left(T(h\circ\eta)^{-1},(h\circ\eta)^{-1}\right)\left([\Gamma(T(h\circ\eta),h\circ\eta)u,\Gamma(T(h\circ\eta),h\circ\eta)v]_{TN}\right),$$

for any $u,v\in\Gamma(TN,\tau_N,N)$, we obtain that

$$\left((TN,\tau_N,N),(T\eta,\eta),[,]_{TN,h}\right)$$

is a generalized Lie algebroid.

For any **Man**-isomorphisms η and h, new and interesting generalized Lie algebroid structures for the tangent vector bundle (TN,τ_N,N) are obtained. For any base $\{t_\alpha,\ \alpha\in\overline{1,m}\}$ of the module of sections $(\Gamma(TN,\tau_N,N),+,\cdot)$ we obtain the structure functions

$$L_{\alpha\beta}^{\gamma}=\left(\theta_{\alpha}^{i}\frac{\partial\theta_{\beta}^{j}}{\partial x^{i}}-\theta_{\beta}^{i}\frac{\partial\theta_{\alpha}^{j}}{\partial x^{i}}\right)\tilde{\theta}_{j}^{\gamma},\ \alpha,\beta,\gamma\in\overline{1,m}$$

where

$$\theta_{\alpha}^{i},\ i,\alpha\in\overline{1,m}$$

are real local functions so that

$$\Gamma(T(h\circ\eta),h\circ\eta)(t_\alpha)=\theta_{\alpha}^{i}\frac{\partial}{\partial x^{i}}$$

and

$$\tilde{\theta}_{j}^{\gamma},\ i,\gamma\in\overline{1,m}$$

are real local functions so that

$$\Gamma\left(T(h\circ\eta)^{-1},(h\circ\eta)^{-1}\right)\left(\frac{\partial}{\partial x^{j}}\right)=\tilde{\theta}_{j}^{\gamma}t_{\gamma}.$$

In particular, using arbitrary isometries (symmetries, translations, rotations,...) for the Euclidean 3-dimensional space Σ, and arbitrary basis for the module of sections we obtain a lot of generalized Lie algebroid structures for the tangent vector bundle $(T\Sigma,\tau_\Sigma,\Sigma)$.

Let $((F,\nu,M),[,]_F,(\rho,Id_M))$ be a Lie algebroid and let $h\in Iso_{\mathbf{Man}}(M,M)$ be. Let $\mathcal{AF}_F$ be a vector fibred $(m+p)$-atlas for the vector bundle (F,ν,M) and let $\mathcal{AF}_{TM}$ be a vector fibred $(m+m)$-atlas for the vector bundle (TM,τ_M,M).

If $(U,\xi_U)\in\mathcal{AF}_{TM}$ and $(V,s_V)\in\mathcal{AF}_F$ such that $U\cap h^{-1}(V)\neq\phi$, then we define the application

$$\begin{array}{ccc}\tau_N^{-1}(U\cap h^{-1}(V))) & \xrightarrow{\bar{\xi}_{U\cap h^{-1}(V)}} & \left(U\cap h^{-1}(V)\right)\times\mathbb{R}^m\\ (\varkappa,u(\varkappa)) & \longmapsto & \left(\varkappa,\xi_{U,\varkappa}^{-1}u(\varkappa)\right).\end{array}$$

Proposition 2.1 *The set*

$$\overline{\mathcal{AF}}_{TM} \overset{put}{=} \bigcup_{\substack{(U,\xi_U)\in\mathcal{AF}_{TM},\ (V,s_V)\in\mathcal{AF}_F \\ U\cap h^{-1}(V)\neq\phi}} \left\{ \left(U\cap h^{-1}(V), \bar{\xi}_{U\cap h^{-1}(V)} \right) \right\}$$

is a vector fibred $m+m$-atlas of the vector bundle (TM,τ_M,M).

If $X = X^{\tilde{\imath}} \frac{\partial}{\partial \varkappa^{\tilde{\imath}}} \in \Gamma(TM,\tau_M,M)$, then we obtain the section

$$\bar{X} = \bar{X}^{\tilde{\imath}} \circ h \frac{\partial}{\partial \bar{\varkappa}^{\tilde{\imath}}} \in \Gamma(TM,\tau_M,M),$$

such that $\bar{X}(\bar{\varkappa}) = X(h(\bar{\varkappa}))$, for any $\bar{\varkappa} \in U\cap h^{-1}(V)$.

The set $\left\{ \frac{\partial}{\partial \bar{\varkappa}^{\tilde{\imath}}},\ \tilde{\imath}\in\overline{1,m} \right\}$ is the natural base of the $\mathcal{F}(M)$-module $(\Gamma(TM,\tau_M,M),+,\cdot)$.

Remark 2.3 If $\left((F,\nu,N), [,]_{F,h}, (\rho,\eta) \right)$ is a generalized Lie algebroid, then we obtain the inclusion $\mathbf{B^v}$-morphism

$$\begin{array}{ccc} \pi^*\ (h^*F) & \hookrightarrow & F \\ h^*\nu\downarrow & & \downarrow\nu \\ E & \xrightarrow{h\circ\pi} & M \end{array} \tag{2.7}$$

3 The Lie algebroid generalized tangent bundle

We consider the following diagram:

$$\begin{array}{ccc} E & & \left(F, [,]_{F,h}, (\rho,\eta) \right) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array} \tag{3.1}$$

where (E,π,M) is a fiber bundle and $\left((F,\nu,M), [,]_{F,h}, (\rho,\eta) \right)$ is a generalized Lie algebroid.

We take (x^i, y^a) as canonical local coordinates on (E,π,M), where $i\in\overline{1,m}$ and $a\in\overline{1,r}$. Let

$$(x^i, y^a) \longrightarrow \left(x^{\acute{\imath}}(x^i), y^{\acute{a}}(x^i, y^a) \right)$$

be a change of coordinates on (E,π,M). Then the coordinates y^a change to $y^{\acute{a}}$ according to the rule:

$$y^{\acute{a}} = \frac{\partial y^{\acute{a}}}{\partial y^a} y^a. \tag{3.2}$$

In particular, if (E,π,M) is vector bundle, then the coordinates y^a change to $y^{\acute{a}}$ according to the rule:

$$y^{\acute{a}} = M_a^{\acute{a}} y^a. \tag{3.2'}$$

Easily we obtain the following

Theorem 3.1 *Let* $\left(\overset{\pi^*\ (h^*F)}{\rho}, Id_E\right)$ *be the* $\mathbf{B^v}$*-morphism of* $(\pi^*\ (h^*F), \pi^*(h^*\nu), M)$ *source and* (TM, τ_M, M) *target, where*

$$
\begin{array}{lccc}
(3.3) \qquad & \pi^*\ (h^*F) & \xrightarrow{\overset{\pi^*\ (h^*F)}{\rho}} & TE \\
& Z^\alpha T_\alpha(u_x) & \longmapsto & \left(Z^\alpha \cdot \rho^i_\alpha \circ h \circ \pi\right) \dfrac{\partial}{\partial x^i}(u_x)
\end{array}
$$

Using the operation

$$
\Gamma(\pi^*\ (h^*F), \pi^*(h^*\nu), M)^2 \xrightarrow{[,]_{\pi^*\ (h^*F)}} \Gamma(\pi^*\ (h^*F), \pi^*(h^*\nu), M)
$$

defined by

$$
\begin{array}{lll}
& [T_\alpha, T_\beta]_{\pi^*\ (h^*F)} & = L^\gamma_{\alpha\beta} \circ h \circ \pi \cdot T_\gamma, \\
(3.4) \qquad & [T_\alpha, fT_\beta]_{\pi^*\ (h^*F)} & = fL^\gamma_{\alpha\beta} \circ h \circ \pi T_\gamma + \rho^i_\alpha \circ h \circ \pi \dfrac{\partial f}{\partial x^i} T_\beta, \\
& [fT_\alpha, T_\beta]_{\pi^*\ (h^*F)} & = -[T_\beta, fT_\alpha]_{\pi^*\ (h^*F)},
\end{array}
$$

for any $f \in \mathcal{F}(E)$, *it results that*

$$
\left((\pi^*\ (h^*F), \pi^*(h^*\nu), M), [,]_{\pi^*\ (h^*F)}, \left(\overset{\pi^*\ (h^*F)}{\rho}, Id_E\right)\right)
$$

is a Lie algebroid which is called the pull-back Lie algebroid of the generalized Lie algebroid $\left((F, \nu, M), [,]_{F,h}, (\rho, \eta)\right)$

If $z = z^\alpha t_\alpha \in \Gamma(F, \nu, M)$, then we obtain the section

$$
Z = (z^\alpha \circ h \circ \pi) T_\alpha \in \Gamma(\pi^*(h^*F), \pi^*(h^*\nu), E)
$$

so that $Z(u_x) = z(h(x))$, for any $u_x \in \pi^{-1}\left(U \cap h^{-1}V\right)$.

Let

$$
\left(\partial_i, \dot{\partial}_a\right) \overset{put}{=} \left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial y^a}\right)
$$

be the base sections for the Lie $\mathcal{F}(E)$-algebra

$$
(\Gamma(TE, \tau_E, E), +, \cdot, [,]_{TE}).
$$

For any sections

$$
Z^\alpha T_\alpha \in \Gamma(\pi^*(h^*F), \pi^*(h^*F), E)
$$

and

$$
Y^a \dot{\partial}_a \in \Gamma(VTE, \tau_E, E)
$$

we obtain the section

$$
\begin{array}{c}
Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a =: Z^\alpha \left(T_\alpha \oplus \left(\rho^i_\alpha \circ h \circ \pi\right) \partial_i\right) + Y^a \left(0_{\pi^*(h^*F)} \oplus \dot{\partial}_a\right) \\
= Z^\alpha T_\alpha \oplus \left(Z^\alpha \left(\rho^i_\alpha \circ h \circ \pi\right) \partial_i + Y^a \dot{\partial}_a\right) \in \Gamma\left(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E\right).
\end{array}
$$

Since we have

$$
\begin{array}{c}
Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a = 0 \\
\Updownarrow \\
Z^\alpha T_\alpha = 0 \ \wedge Z^\alpha \left(\rho^i_\alpha \circ h \circ \pi\right) \partial_i + Y^a \dot{\partial}_a = 0,
\end{array}
$$

it implies $Z^{\alpha}=0,\ \alpha\in\overline{1,p}$ and $Y^{a}=0,\ a\in\overline{1,r}$.

Therefore, the sections $\tilde{\partial}_1,...,\tilde{\partial}_p,\dot{\tilde{\partial}}_1,...,\dot{\tilde{\partial}}_r$ are linearly independent.

We consider the vector subbundle $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ of the vector bundle $\left(\pi^*(h^*F)\oplus TE,\overset{\oplus}{\pi},E\right)$, for which the $\mathcal{F}(E)$-module of sections is the $\mathcal{F}(E)$-submodule of $\left(\Gamma\left(\pi^*(h^*F)\oplus TE,\overset{\oplus}{\pi},E\right),+,\cdot\right)$, generated by the set of sections $\left(\tilde{\partial}_\alpha,\dot{\tilde{\partial}}_a\right)$.

The base sections $\left(\tilde{\partial}_\alpha,\dot{\tilde{\partial}}_a\right)$ will be called the *natural* (ρ,η)*-base.*

The matrix of coordinate transformation on $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ at a change of fibred charts is

$$\left\|\begin{array}{cc}\Lambda_{\alpha}^{\acute{\alpha}}\circ h\circ\pi & 0\\ \left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\dfrac{\partial y^{\acute{a}}}{\partial x^{i}} & \dfrac{\partial y^{\acute{a}}}{\partial y^{a}}\end{array}\right\|. \tag{3.5}$$

In particular, if (E,π,M) is a vector bundle, then the matrix of coordinate transformation on $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ at a change of fibred charts is

$$\left\|\begin{array}{cc}\Lambda_{\alpha}^{\acute{\alpha}}\circ h\circ\pi & 0\\ \left(\rho_{a}^{i}\circ h\circ\pi\right)\dfrac{\partial M_{b}^{\acute{a}}\circ\pi}{\partial x_{i}}y^{b} & M_{a}^{\acute{a}}\circ\pi\end{array}\right\|. \tag{3.6}$$

Easily we obtain

Theorem 3.1 *Let* $(\tilde{\rho},Id_E)$ *be the* $\mathbf{B^v}$*-morphism of* $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ *source and* (TE,τ_E,E) *target, where*

$$\begin{array}{c}(\rho,\eta)\,TE\xrightarrow{\tilde{\rho}}TE\\ \left(Z^{\alpha}\tilde{\partial}_{\alpha}+Y^{a}\dot{\tilde{\partial}}_{a}\right)(u_x)\longmapsto\left(Z^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\partial_i+Y^{a}\dot{\partial}_a\right)(u_x)\end{array} \tag{3.7}$$

Using the operation

$$\Gamma((\rho,\eta)TE,(\rho,\eta)\tau_E,E)^2\xrightarrow{[,]_{(\rho,\eta)TE}}\Gamma((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$$

defined by

$$\begin{array}{l}\left[Z_1^{\alpha}\tilde{\partial}_{\alpha}+Y_1^{a}\dot{\tilde{\partial}}_{a},Z_2^{\beta}\tilde{\partial}_{\beta}+Y_2^{b}\dot{\tilde{\partial}}_{b}\right]_{(\rho,\eta)TE}\\ =\left[Z_1^{\alpha}T_{\alpha},Z_2^{\beta}T_{\beta}\right]_{\pi^*(h^*F)}\oplus\left[Z_1^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\partial_i+Y_1^{a}\dot{\partial}_a,\right.\\ \qquad\left.Z_2^{\beta}\left(\rho_{\beta}^{j}\circ h\circ\pi\right)\partial_j+Y_2^{b}\dot{\partial}_b\right]_{TE},\end{array} \tag{3.8}$$

for any $Z_1^{\alpha}\tilde{\partial}_{\alpha}+Y_1^{a}\dot{\tilde{\partial}}_{a}$ *and* $Z_2^{\beta}\tilde{\partial}_{\beta}+Y_2^{b}\dot{\tilde{\partial}}_{b}$*, we obtain that the couple* $\left([,]_{(\rho,\eta)TE},(\tilde{\rho},Id_E)\right)$ *is a Lie algebroid structure for the vector bundle* $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$.

Remark 3.2 In particular, if $h=Id_M$, then the Lie algebroid

$$\left(((Id_{TM},Id_M)TE,(Id_{TM},Id_M)\tau_E,E),[,]_{(Id_{TM},Id_M)TE},\left(\widetilde{Id_{TM}},Id_E\right)\right)$$

is isomorphic with the usual Lie algebroid

$$\left(\left(TE,\tau_E,E\right),[,]_{TE},\left(Id_{TE},Id_E\right)\right).$$

This is a reason for which the Lie algebroid

$$\left(\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right),[,]_{\left(\rho,\eta\right)TE},\left(\tilde{\rho},Id_E\right)\right)\ ,$$

will be called the *Lie algebroid generalized tangent bundle.*

The vector bundle $\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$ will be called the *generalized tangent bundle.*

3.1 The generalized tangent bundle of dual vector bundle

Let (E,π,M) be a vector bundle. We build the generalized tangent bundle of dual vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ using the diagram:

$$(3.1.1)\qquad \begin{array}{ccc} \overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right) \\ \overset{*}{\pi}\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & N \end{array}\ ,$$

where $\left(\left(F,\nu,N\right),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

We take $\left(x^i,p_a\right)$ as canonical local coordinates on $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, where $i\in\overline{1,m}$ and $a\in\overline{1,r}$.

Consider

$$\left(x^i,p_a\right)\longrightarrow\left(x^{\acute{i}}\left(x^i\right),p_{\acute{a}}\left(x^i,p_a\right)\right)$$

a change of coordinates on $\left(\overset{*}{E},\overset{*}{\pi},M\right)$. Then the coordinates p_a change to $p_{\acute{a}}$ according to the rule:

$$(3.1.2)\qquad p_{\acute{a}}=M_{\acute{a}}^{a}p_a.$$

Let

$$\left(\overset{*}{\partial}_i,\dot{\partial}^a\right)\overset{put}{=}\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial p_a}\right)$$

be the base sections for the Lie $\mathcal{F}\left(\overset{*}{E}\right)$-algebra

$$\left(\Gamma\left(T\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot,[,]_{T\overset{*}{E}}\right).$$

For any sections

$$Z^{\alpha}T_{\alpha}\in\Gamma\left(\overset{**}{\pi}\left(h^*F\right),\overset{**}{\pi}\left(h^*\nu\right),\overset{*}{E}\right)$$

and

$$Y_a\dot{\partial}^a\in\Gamma\left(VT\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right),$$

we obtain the section

$$\begin{aligned}Z^{\alpha}\tilde{\overset{*}{\partial}}_{\alpha}+Y_{a}\tilde{\overset{\cdot}{\partial}}^{a} &=: Z^{\alpha}\left(T_{\alpha}\oplus\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\partial}_{i}\right)+Y_{a}\left(0_{\overset{*}{\pi}^{*}(h^{*}F)}\oplus\dot{\partial}^{a}\right)\\ &= Z^{\alpha}T_{\alpha}\oplus\left(Z^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\partial}_{i}+Y_{a}\dot{\partial}^{a}\right)\in\Gamma\left(\overset{*}{\pi}^{*}(h^{*}F)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}\right).\end{aligned}$$

Since we have

$$\begin{gathered}Z^{\alpha}\tilde{\overset{*}{\partial}}_{\alpha}+Y_{a}\tilde{\overset{\cdot}{\partial}}^{a}=0_{\overset{*}{\pi}^{*}(h^{*}F)\oplus T\overset{*}{E}}\\ \Updownarrow\\ Z^{\alpha}T_{\alpha}=0_{\overset{*}{\pi}^{*}(h^{*}F)}\ \wedge Z^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\partial}_{i}+Y_{a}\dot{\partial}^{a}=0_{T\overset{*}{E}},\end{gathered}$$

it implies $Z^{\alpha}=0,\ \alpha\in\overline{1,p}$ and $Y_{a}=0,\ a\in\overline{1,r}$.

Therefore, the sections

$$\tilde{\overset{*}{\partial}}_{1},...,\tilde{\overset{*}{\partial}}_{p},\tilde{\overset{\cdot}{\partial}}^{1},...,\tilde{\overset{\cdot}{\partial}}^{r}$$

are linearly independent.

We consider the vector subbundle

$$\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

of vector bundle

$$\left(\overset{*}{\pi}^{*}(h^{*}F)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}\right),$$

for which the $\mathcal{F}\left(\overset{*}{E}\right)$-module of sections is the $\mathcal{F}\left(\overset{*}{E}\right)$-submodule of

$$\left(\Gamma\left(\overset{*}{\pi}^{*}(h^{*}F)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}\right),+,\cdot\right),$$

generated by the family of sections $\left(\tilde{\overset{*}{\partial}}_{\alpha},\tilde{\overset{\cdot}{\partial}}^{a}\right)$ which is called the *natural* (ρ,η)*-base.*

The matrix of coordinate transformation on $\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$ at a change of fibred charts is

$$\left\Vert\begin{array}{cc}\Lambda_{\alpha}^{\acute{\alpha}}\circ h\circ\overset{*}{\pi} & 0\\ \left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\dfrac{\partial M_{\acute{a}}^{b}\circ\overset{*}{\pi}}{\partial x_{i}}p_{b} & M_{\acute{a}}^{a}\circ\overset{*}{\pi}\end{array}\right\Vert. \tag{3.1.3}$$

We consider the operation $[,]_{(\rho,\eta)T\overset{*}{E}}$ defined by

$$\begin{aligned}&\left[Z_{1}^{\alpha}\tilde{\overset{*}{\partial}}_{\alpha}+Y_{a}^{1}\tilde{\overset{\cdot}{\partial}}^{a},Z_{2}^{\beta}\tilde{\overset{*}{\partial}}_{\beta}+Y_{b}^{2}\tilde{\overset{\cdot}{\partial}}^{b}\right]_{(\rho,\eta)T\overset{*}{E}}=\\ &=\left[Z_{1}^{\alpha}T_{\alpha},Z_{2}^{\beta}T_{\beta}\right]_{\overset{*}{\pi}^{*}(h^{*}F)}\oplus\Big[Z_{1}^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\partial}_{i}+Y_{a}^{1}\dot{\partial}^{a},\\ &\qquad Z_{2}^{\beta}\left(\rho_{\beta}^{j}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\partial}_{j}+Y_{b}^{2}\dot{\partial}^{b}\Big]_{T\overset{*}{E}},\end{aligned} \tag{3.1.4}$$

for any sections $Z_1^{\alpha}\overset{*}{\tilde{\partial}}_{\alpha}+Y_a^1\overset{\cdot}{\tilde{\partial}}^{a}$ and $Z_2^{\beta}\overset{*}{\tilde{\partial}}_{\beta}+Y_b^2\overset{\cdot}{\tilde{\partial}}^{b}$.

Let $\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)$ be the $\mathbf{B^v}$-morphism of $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ source and $\left(T\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right)$ target, where

$$
\begin{array}{c}
(\rho,\eta)\,T\overset{*}{E}\ \xrightarrow{\overset{*}{\tilde{\rho}}}\ T\overset{*}{E}\\
\left(Z^{\alpha}\overset{*}{\tilde{\partial}}_{\alpha}+Y_a\overset{\cdot}{\tilde{\partial}}^{a}\right)(\overset{*}{u}_x)\longmapsto\left(Z^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\partial}_i+Y_a\dot{\partial}^{a}\right)(\overset{*}{u}_x)
\end{array}
\tag{3.1.5}
$$

The Lie algebroid generalized tangent bundle of the dual vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ will be denoted

$$
\left(\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),[,]_{(\rho,\eta)T\overset{*}{E}},\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\right).
$$

4 (Linear) (ρ,η)-connections

We consider the diagram:

$$
\begin{array}{ccc}
E & & \left(F,[,]_{F,h},(\rho,\eta)\right)\\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\nu}\\
M & \xrightarrow{h} & N
\end{array}
$$

where $(E,\pi,M)\in|\mathbf{B}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Let

$$
\left(((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E),[,]_{(\rho,\eta)TE},(\tilde{\rho},Id_E)\right)
$$

be the Lie algebroid generalized tangent bundle of the fiber bundle (E,π,M).

We consider the $\mathbf{B^v}$-morphism $((\rho,\eta)\,\pi!,Id_E)$ given by the commutative diagram

$$
\begin{array}{ccc}
(\rho,\eta)\,TE & \xrightarrow{(\rho,\eta)\pi!} & \pi^*(h^*F)\\
{\scriptstyle(\rho,\eta)\tau_E}\downarrow & & \downarrow{\scriptstyle pr_1}\\
E & \xrightarrow{id_E} & E
\end{array}
\tag{4.1}
$$

This is defined as:

$$
(\rho,\eta)\,\pi!\left(\left(Z^{\alpha}\overset{*}{\tilde{\partial}}_{\alpha}+Y^{a}\dot{\tilde{\partial}}_{a}\right)(u_x)\right)=(Z^{\alpha}T_{\alpha})(u_x),
\tag{4.2}
$$

for any $Z^{\alpha}\overset{*}{\tilde{\partial}}_{\alpha}+Y^{a}\dot{\tilde{\partial}}_{a}\in\Gamma((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$.

Using the $\mathbf{B^v}$-morphism $((\rho,\eta)\,\pi!,Id_E)$, and the the $\mathbf{B^v}$-morphism (2.7) we obtain the *tangent* (ρ,η)-*application* $((\rho,\eta)\,T\pi,h\circ\pi)$ of $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ source and (F,ν,N) target.

Definition 4.1 The kernel of the tangent (ρ,η)-application is writen

$$(V(\rho,\eta)TE,(\rho,\eta)\tau_E,E)$$

and it is called *the vertical subbundle.*

We remark that the set $\left\{\dot{\tilde{\partial}}_a,\ a\in\overline{1,r}\right\}$ is a base of the $\mathcal{F}(E)$-module

$$(\Gamma(V(\rho,\eta)TE,(\rho,\eta)\tau_E,E),+,\cdot).$$

Proposition 4.1 *The short sequence of vector bundles*

$$\begin{tikzcd} 0 \arrow[r, hook, "i"] \arrow[d] & V(\rho,\eta)TE \arrow[r, hook, "i"] \arrow[d] & (\rho,\eta)TE \arrow[r, "(\rho,\eta)\pi!"] \arrow[d] & \pi^*(h^*F) \arrow[r] \arrow[d] & 0 \arrow[d] \\ E \arrow[r, "Id_E"] & E \arrow[r, "Id_E"] & E \arrow[r, "Id_E"] & E \arrow[r, "Id_E"] & E \end{tikzcd} \tag{4.3}$$

is exact.

Definition 4.2 *A* **Man**-morphism $(\rho,\eta)\Gamma$ of $(\rho,\eta)TE$ source and $V(\rho,\eta)TE$ target defined by

$$(\rho,\eta)\Gamma\left(Z^\gamma\overset{*}{\tilde{\partial}}_\gamma+Y^a\dot{\tilde{\partial}}_a\right)(u_x)=\left(Y^a+(\rho,\eta)\Gamma^a_\gamma Z^\gamma\right)\dot{\tilde{\partial}}_a(u_x), \tag{4.4}$$

so that the $\mathbf{B^v}$-morphism $((\rho,\eta)\Gamma,Id_E)$ is a split to the left in the previous exact sequence, will be called (ρ,η)-*connection for the fiber bundle* (E,π,M).

The (ρ,Id_M)-connection will be called ρ-*connection* and will be denoted $\rho\Gamma$ and the (Id_{TM},Id_M)-connection will be called *connection* and will be denoted Γ.

Definition 4.3 If $(\rho,\eta)\Gamma$ is a (ρ,η)-connection for the fiber bundle (E,π,M), then the kernel of the $\mathbf{B^v}$-morphism $((\rho,\eta)\Gamma,Id_E)$ is written $(H(\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ and will be called the *horizontal vector subbundle.*

Definition 4.4 If $(E,\pi,M)\in|\mathbf{B}|$, then the **B**-morphism (Π,π) defined by the commutative diagram

$$\begin{tikzcd} V(\rho,\eta)TE \arrow[r, "\Pi"] \arrow[d, "(\rho,\eta)\tau_E"'] & E \arrow[d, "\pi"] \\ E \arrow[r, "\pi"] & M \end{tikzcd} \tag{4.5}$$

so that the components of the image of the vector $Y^a\dot{\tilde{\partial}}_a(u_x)$ are the real numbers $Y^1(u_x),...,Y^r(u_x)$ will be called the *canonical projection* **B**-*morphism.*

In particular, if $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left\{s_a,a\in\overline{1,r}\right\}$ is a base of the $\mathcal{F}(M)$-module of sections $(\Gamma(E,\pi,M),+,\cdot)$, then Π is defined by

$$\Pi\left(Y^a\dot{\tilde{\partial}}_a(u_x)\right)=Y^a(u_x)s_a(\pi(u_x))=Y^a(u_x)s_a(x). \tag{4.6}$$

Theorem 4.1 *If* $(\rho,\eta)\Gamma$ *is a* (ρ,η)-*connection for the fiber bundle* (E,π,M), *then its components satisfy the law of transformation*

$$(\rho,\eta)\Gamma^{a'}_{\gamma'}=\frac{\partial y^{a'}}{\partial y^a}\left[\rho^k_\gamma\circ(h\circ\pi)\frac{\partial y^a}{\partial x^k}+(\rho,\eta)\Gamma^a_\gamma\right]\Lambda^\gamma_{\gamma'}\circ(h\circ\pi). \tag{4.7}$$

If $(\rho,\eta)\,\Gamma$ is a (ρ,η)-connection for the vector bundle (E,π,M), then its components satisfy the law of transformation

$$(\rho,\eta)\,\Gamma_{\gamma'}^{a'}{=}M_{a}^{a'}{\circ}\pi\Big[\rho_{\gamma}^{k}{\circ}\,(h{\circ}\pi)\tfrac{\partial M_{b'}^{a}\circ\pi}{\partial x^{k}}y^{b'}{+}\,(\rho,\eta)\Gamma_{\gamma}^{a}\Big]\Lambda_{\gamma'}^{\gamma}{\circ}\,(h{\circ}\pi)\,. \tag{4.7'}$$

In the particular case of Lie algebroids, $(\eta,h)=(Id_M,Id_M)$, the relations (4.7′) become

$$\rho\Gamma_{\gamma'}^{a'}=M_{a}^{a'}\circ\pi\left[\rho_{\gamma}^{k}\circ\pi\tfrac{\partial M_{b'}^{a}\circ\pi}{\partial x^{k}}y^{b'}+\rho\Gamma_{\gamma}^{a}\right]\Lambda_{\gamma'}^{\gamma}\circ\pi. \tag{4.7''}$$

In the classical case, $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, the relations (4.7″) become

$$\Gamma_{k'}^{i'}=\tfrac{\partial x^{i'}}{\partial x^{i}}\circ\tau_M\left[\tfrac{\partial}{\partial x^{k}}\left(\tfrac{\partial x^{i}}{\partial x^{j'}}\circ\tau_M\right)y^{j'}+\Gamma_{k}^{i}\right]\tfrac{\partial x^{k}}{\partial x^{k'}}\circ\tau_M. \tag{4.7'''}$$

Proof. Let (Π,π) be the canonical projection $\mathbf{B}$-morphism.

Obviously, the components of

$$\Pi\circ(\rho,\eta)\,\Gamma\left(Z^{\gamma'}\overset{*}{\tilde{\partial}}_{\gamma'}+Y^{a'}\dot{\tilde{\partial}}_{a'}\right)(u_x)$$

are the real numbers

$$\left(Y^{a'}+(\rho,\eta)\,\Gamma_{\gamma'}^{a'}Z^{\gamma'}\right)(u_x)\,.$$

Since

$$\begin{aligned}\left(Z^{\gamma'}\overset{*}{\tilde{\partial}}_{\gamma'}+Y^{a'}\dot{\tilde{\partial}}_{a'}\right)(u_x)&=Z^{\gamma'}\Lambda_{\gamma'}^{\gamma}\circ h\circ\pi\overset{*}{\tilde{\partial}}_{\gamma}(u_x)\\&+\left(Z^{\gamma'}\rho_{\gamma'}^{i'}\circ h\circ\pi\frac{\partial y^{a}}{\partial x^{i'}}+\frac{\partial y^{a}}{\partial y^{a'}}Y^{a'}\right)\dot{\tilde{\partial}}_{a}(u_x)\,,\end{aligned}$$

it results that the components of

$$\Pi\circ(\rho,\eta)\,\Gamma\left(Z^{\gamma'}\overset{*}{\tilde{\partial}}_{\gamma'}+Y^{a'}\dot{\tilde{\partial}}_{a'}\right)(u_x)$$

are the real numbers

$$\left(Z^{\gamma'}\rho_{\gamma'}^{i'}\circ h\circ\pi\tfrac{\partial y^{a}}{\partial x^{i'}}+\tfrac{\partial y^{a}}{\partial y^{a'}}Y^{a'}+(\rho,\eta)\,\Gamma_{\gamma}^{a}Z^{\gamma'}\Lambda_{\gamma'}^{\gamma}\circ h\circ\pi\right)(u_x)\,\tfrac{\partial y^{a'}}{\partial y^{a}},$$

where

$$\left\|\tfrac{\partial y^{a}}{\partial y^{a'}}\right\|=\left\|\tfrac{\partial y^{a'}}{\partial y^{a}}\right\|^{-1}.$$

Therefore, we have:

$$\left(Z^{\gamma'}\rho_{\gamma'}^{i'}\circ h\circ\pi\tfrac{\partial y^{a}}{\partial x^{i'}}+\tfrac{\partial y^{a}}{\partial y^{a'}}Y^{a'}+(\rho,\eta)\,\Gamma_{\gamma}^{a}Z^{\gamma'}\Lambda_{v'}^{\gamma}\circ h\circ\pi\right)\tfrac{\partial y^{a'}}{\partial y^{a}}=Y^{a'}+(\rho,\eta)\,\Gamma_{\gamma'}^{a'}Z^{\gamma'}.$$

After some calculations we obtain:

$$(\rho,\eta)\,\Gamma_{\gamma'}^{a'}=\tfrac{\partial y^{a'}}{\partial y^{a}}\left(\rho_{\gamma}^{i}\circ(h\circ\pi)\,\tfrac{\partial y^{a}}{\partial x^{i}}+(\rho,\eta)\,\Gamma_{\gamma}^{a}\right)\Lambda_{\gamma'}^{\gamma}\circ h\circ\pi. \qquad q.e.d.$$

Remark 4.1 If Γ is a Ehresmann connection for the vector bundle (E,π,M) on components Γ_{k}^{a}, then the differentiable real local functions $(\rho,\eta)\,\Gamma_{\gamma}^{a}=\left(\rho_{\gamma}^{k}\circ h\circ\pi\right)\Gamma_{k}^{a}$

are the components of a (ρ,η)-connection $(\rho,\eta)\,\Gamma$ for the vector bundle (E,π,M). This (ρ,η)-connection will be called the *(ρ,η)-connection associated to the connection* Γ.

Definition 4.5 If $(\rho,\eta)\,\Gamma$ is a (ρ,η)-connection for the vector bundle (E,π,M) and $z=z^{\gamma}t_{\gamma}\in\Gamma(F,\nu,M)$, then the application

$$\begin{array}{ccc}\Gamma(E,\pi,M) & \xrightarrow{(\rho,\eta)D_z} & \Gamma(E,\pi,M)\\ u=u^a s_a & \longmapsto & (\rho,\eta)\,D_z u\end{array}$$

where

$$(\rho,\eta)\,D_z u=z^{\gamma}\circ h\left(\rho_{\gamma}^{k}\circ h\tfrac{\partial u^a}{\partial x^k}+(\rho,\eta)\,\Gamma_{\gamma}^{a}\circ u\right)s_a \tag{4.8}$$

will be called the *covariant (ρ,η)-derivative associated to (ρ,η)-connection $(\rho,\eta)\,\Gamma$ with respect to the section z.*

In the particular case of Lie algebroids, $(\eta,h)=(Id_M,Id_M)$, the relations (4.8) become

$$\rho D_z u=z^{\gamma}\left(\rho_{\gamma}^{k}\tfrac{\partial u^a}{\partial x^k}+\rho\Gamma_{\gamma}^{a}\circ u\right)s_a. \tag{4.8$'$}$$

In the classical case, $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, the relations (4.8$'$) become

$$D_X Y=X^k\left(\tfrac{\partial Y^i}{\partial x^k}+\Gamma_k^i\circ Y\right)\partial_i. \tag{4.8$''$}$$

Definition 4.6 Let $(\rho,\eta)\,\Gamma$ be a (ρ,η)-connection for the fiber bundle (E,π,M). If for each local vector $(m+r)$-chart (U,s_U) and for each local vector $(n+p)$-chart (V,t_V) so that $U\cap h^{-1}(V)\neq\phi$, it exists the differentiable real functions $(\rho,\eta)\,\Gamma_{b\gamma}^{a}$ defined on $U\cap h^{-1}(V)$ such that

$$(\rho,\eta)\,\Gamma_{\gamma}^{a}\circ u=(\rho,\eta)\,\Gamma_{b\gamma}^{a}\cdot u^b,\forall u=u^b s_b\in\Gamma(E,\pi,M), \tag{4.9}$$

then we say that $(\rho,\eta)\,\Gamma$ *is linear.*

The differentiable real local functions $(\rho,\eta)\,\Gamma_{b\gamma}^{a}$ will be called the *Christoffel coefficients of linear (ρ,η)-connection $(\rho,\eta)\,\Gamma$.*

Proposition 4.2 *If $(\rho,\eta)\,\Gamma$ is a linear (ρ,η)-connection for the fiber bundle (E,π,M), then its components satisfy the law of transformation*

$$(\rho,\eta)\Gamma_{b'\gamma'}^{a'}=\tfrac{\partial y^{a'}}{\partial y^a}\left[\rho_{\gamma}^{k}\circ h\tfrac{\partial}{\partial x^k}\left(\tfrac{\partial y^a}{\partial y^{b'}}\right)+(\rho,\eta)\,\Gamma_{b\gamma}^{a}\tfrac{\partial y^b}{\partial y^{b'}}\right]\Lambda_{\gamma'}^{\gamma}\circ h. \tag{4.10}$$

If $(\rho,\eta)\,\Gamma$ is a linear (ρ,η)-connection for the vector bundle (E,π,M), then its components satisfy the law of transformation

$$(\rho,\eta)\Gamma_{b'\gamma'}^{a'}=M_a^{a'}\left[\rho_{\gamma}^{k}\circ h\tfrac{\partial M_{b'}^{a}}{\partial x^k}+(\rho,\eta)\Gamma_{b\gamma}^{a}M_{b'}^{b}\right]\Lambda_{\gamma'}^{\gamma}\circ h. \tag{4.10$'$}$$

In the particular case of Lie algebroids, $(\eta,h)=(Id_M,Id_M)$, the relations (4.10$'$) *become*

$$\rho\Gamma_{b'\gamma'}^{a'}=M_a^{a'}\left[\rho_{\gamma}^{k}\tfrac{\partial M_{b'}^{a}}{\partial x^k}+\rho\Gamma_{b\gamma}^{a}M_{b'}^{b}\right]\Lambda_{\gamma'}^{\gamma}. \tag{4.10$''$}$$

In the classical case, $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, the relations (4.10$''$) *become*

$$\Gamma_{j'k'}^{i'}=\tfrac{\partial x^{i'}}{\partial x^i}\left[\tfrac{\partial}{\partial x^k}\left(\tfrac{\partial x^i}{\partial x^{j'}}\right)+\Gamma_{jk}^{i}\tfrac{\partial x^j}{\partial x^{j'}}\right]\tfrac{\partial x^k}{\partial x^{k'}}. \tag{4.10$'''$}$$

Remark 4.2 If $(\rho,\eta)\,\Gamma$ is a linear (ρ,η)-connection for the vector bundle (E,π,M), then, for any $z=z^{\gamma}t_{\gamma}\in\Gamma\,(F,\nu,M)$, we obtain the *covariant (ρ,η)-derivative associated to the linear (ρ,η)-connection $(\rho,\eta)\,\Gamma$ with respect to the section z*

$$\begin{array}{ccc}\Gamma\,(E,\pi,M) & \xrightarrow{(\rho,\eta)D_z} & \Gamma\,(E,\pi,M)\\ u{=}u^a s_a & \longmapsto & (\rho,\eta)D_z u\end{array}$$

defined by

$$(\rho,\eta)D_z u{=}z^{\gamma}{\circ}h\left(\rho_{\gamma}^{k}{\circ}h\frac{\partial u^a}{\partial x^k}{+}(\rho,\eta)\Gamma_{b\gamma}^{a}\cdot u^b\right)s_a. \tag{4.11}$$

In the particular case of Lie algebroids, $(\eta,h)=(Id_M,Id_M)$, the relations (4.11) become

$$\rho D_z u{=}z^{\gamma}\left(\rho_{\gamma}^{k}\frac{\partial u^a}{\partial x^k}{+}\rho\Gamma_{b\gamma}^{a}\cdot u^b\right)s_a. \tag{4.11$'$}$$

In the classical case, $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, the relations (4.11$'$) become

$$D_X Y{=}X^k\left(\frac{\partial Y^i}{\partial x^k}{+}\Gamma_{jk}^{i}\cdot Y^j\right)\partial_i. \tag{4.11$''$}$$

4.1 (Linear) (ρ,η)-connections for dual vector bundle

Let (E,π,M) be a vector bundle.

We consider the following diagram:

$$\begin{array}{ccc}\overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right)\\ \overset{*}{\pi}\downarrow & & \downarrow\nu\\ M & \xrightarrow{\ \ h\ \ } & N\end{array}\quad, \tag{4.1.1}$$

where $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Let

$$\left(\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),[,]_{(\rho,\eta)T\overset{*}{E}},\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\right)$$

be the Lie algebroid generalized tangent bundle of the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

We consider the $\mathbf{B^v}$-morphism $\left((\rho,\eta)\,\overset{*}{\pi}!,Id_{\overset{*}{E}}\right)$ given by the commutative diagram

$$\begin{array}{ccc}(\rho,\eta)\,T\overset{*}{E} & \xrightarrow{(\rho,\eta)\overset{*}{\pi}!} & \overset{*}{\pi}{}^{*}\,(h^*F)\\ {\scriptstyle(\rho,\eta)\tau_{\overset{*}{E}}}\downarrow & & \downarrow{\scriptstyle pr_1}\\ \overset{*}{E} & \xrightarrow{\ id_{\overset{*}{E}}\ } & \overset{*}{E}\end{array} \tag{4.1.2}$$

Using the components, this is defined as:

$$(\rho,\eta)\,\overset{*}{\pi}!\left(Z^{\alpha}\overset{*}{\tilde{\partial}}_{\alpha}+Y_a\overset{\cdot}{\tilde{\partial}}{}^{a}\right)\left(\overset{*}{u}_x\right)=\left(Z^{\alpha}T_{\alpha}\right)\left(\overset{*}{u}_x\right), \tag{4.1.3}$$

for any $Z^{\alpha}\overset{*}{\tilde{\partial}}_{\alpha}+Y_{a}\overset{\cdot\, a}{\tilde{\partial}} \in \left((\rho,\eta)\, T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Using the $\mathbf{B^v}$-morphism $\left((\rho,\eta)\overset{*}{\pi}!, Id_{\overset{*}{E}}\right)$ and the $\mathbf{B^v}$-morphism (2.7) we obtain the *tangent* (ρ,η)*-application* $\left((\rho,\eta)\, T\overset{*}{\pi}, h\circ\overset{*}{\pi}\right)$ of $\left((\rho,\eta)\, T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ source and (F,ν,N) target.

Definition 4.1.1 The kernel of the tangent (ρ,η)-application

$$\left((\rho,\eta)\, T\overset{*}{\pi}, h\circ\overset{*}{\pi}\right)$$

is written as

$$\left(V\,(\rho,\eta)\, T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

and will be called the *vertical subbundle.*

The set $\left\{\overset{\cdot\, a}{\tilde{\partial}},\ a\in\overline{1,r}\right\}$ is a base for the $\mathcal{F}\left(\overset{*}{E}\right)$-module

$$\left(\Gamma\left(V\,(\rho,\eta)\, T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right).$$

Proposition 4.1.1 *The short sequence of vector bundles*

$$\text{(4.1.4)}\qquad \begin{array}{ccccccccc}
0 & \overset{i}{\hookrightarrow} & V(\rho,\eta)T\overset{*}{E} & \overset{i}{\hookrightarrow} & (\rho,\eta)T\overset{*}{E} & \overset{(\rho,\eta)\overset{*}{\pi}!}{\longrightarrow} & \overset{*}{\pi}{}^{*}\,(h^{*}F) & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
\overset{*}{E} & \xrightarrow{Id_{\overset{*}{E}}} & \overset{*}{E} & \xrightarrow{Id_{\overset{*}{E}}} & \overset{*}{E} & \xrightarrow{Id_{\overset{*}{E}}} & \overset{*}{E} & \xrightarrow{Id_{\overset{*}{E}}} & \overset{*}{E}
\end{array}$$

is exact.

Definition 4.1.2 *A* **Man**-morphism $(\rho,\eta)\,\Gamma$ of $(\rho,\eta)\, T\overset{*}{E}$ source and $V\,(\rho,\eta)\, T\overset{*}{E}$ target defined by

$$\text{(4.1.5)}\qquad (\rho,\eta)\,\Gamma\left(Z^{\gamma}\overset{*}{\tilde{\partial}}_{\gamma}+Y_{a}\overset{\cdot\, a}{\tilde{\partial}}\right)\left(\overset{*}{u}_{x}\right)=\left(Y_{b}-(\rho,\eta)\,\Gamma_{b\gamma}Z^{\gamma}\right)\overset{\cdot\, b}{\tilde{\partial}}\left(\overset{*}{u}_{x}\right),$$

such that the $\mathbf{B^v}$-morphism $\left((\rho,\eta)\,\Gamma, Id_{\overset{*}{E}}\right)$ is a split to the left in the previous exact sequence, will be called (ρ,η)*-connection for the dual vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

The differentiable real local functions $(\rho,\eta)\,\Gamma_{b\gamma}$ will be called the *components of* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$.

The (ρ, Id_{M})-connection for the dual vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ will be called ρ*-connection for the dual vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and will be denoted $\rho\Gamma$.

The (Id_{TM}, Id_{M})-connection for the dual vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ will be called *connection for the dual vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and will be denoted Γ.

Let $\left\{s^a,\ a\in\overline{1,r}\right\}$ be the dual base of the base $\left\{s_a, a\in\overline{1,r}\right\}$.

The $\mathbf{B^v}$-morphism $\left(\overset{*}{\Pi},\overset{*}{\pi}\right)$ defined by the commutative diagram

$$
\begin{tikzcd}
V\left(\rho,\eta\right)T\overset{*}{E} \arrow[r, "\overset{*}{\Pi}"] \arrow[d, "\left(\rho,\eta\right)\tau_{\overset{*}{E}}"'] & \overset{*}{E} \arrow[d, "\overset{*}{\pi}"] \\
\overset{*}{E} \arrow[r, "\overset{*}{\pi}"] & M
\end{tikzcd}
,\tag{4.1.6}
$$

where, $\overset{*}{\Pi}$ is defined by

$$
\overset{*}{\Pi}\left(Y_a\overset{\cdot\ a}{\tilde{\partial}}\left(\overset{*}{u}_x\right)\right)=Y_a\left(\overset{*}{u}_x\right)s^a\left(\overset{*}{\pi}\left(\overset{*}{u}_x\right)\right),\tag{4.1.7}
$$

is canonical projection $\mathbf{B^v}$-morphism.

Theorem 4.1.1 *If $\left(\rho,\eta\right)\Gamma$ is a $\left(\rho,\eta\right)$-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then its components satisfy the law of transformation*

$$
\left(\rho,\eta\right)\Gamma_{b'\gamma'}=M_{b'}^{b}\circ\overset{*}{\pi}\left[-\rho_{\gamma}^{k}\circ h\circ\overset{*}{\pi}\cdot\frac{\partial M_{b}^{a'}\circ\overset{*}{\pi}}{\partial x^k}p_{a'}+\left(\rho,\eta\right)\Gamma_{b\gamma}\right]\Lambda_{\gamma'}^{\gamma}\circ\left(h\circ\overset{*}{\pi}\right).\tag{4.1.8}
$$

In the particular case of Lie algebroids, $\left(\eta,h\right)=\left(Id_M,Id_M\right)$, the relations (4.1.8) *become*

$$
\rho\Gamma_{b'\gamma'}=M_{b'}^{b}\circ\overset{*}{\pi}\left[-\rho_{\gamma}^{k}\circ\overset{*}{\pi}\cdot\frac{\partial M_{b}^{a'}\circ\overset{*}{\pi}}{\partial x^k}p_{a'}+\rho\Gamma_{b\gamma}\right]\Lambda_{\gamma'}^{\gamma}\circ\overset{*}{\pi}.\tag{4.1.8'}
$$

In the classical case, $\left(\rho,\eta,h\right)=\left(Id_{TM},Id_M,Id_M\right)$, the relations (4.1.8′) *become*

$$
\Gamma_{j'k'}=\tfrac{\partial x^j}{\partial x^{j'}}\circ\overset{*}{\tau}_M\left[-\tfrac{\partial}{\partial x^k}\left(\tfrac{\partial x^{i'}}{\partial x^j}\circ\overset{*}{\tau}_M\right)p_{i'}+\Gamma_{jk}\right]\tfrac{\partial x^k}{\partial x^{k'}}\circ\overset{*}{\tau}_M.\tag{4.1.8''}
$$

Proof. Let $\left(\overset{*}{\Pi},\overset{*}{\pi}\right)$ be the canonical projection $\mathbf{B}$-morphism.

Obviously, the components of

$$
\overset{*}{\Pi}\circ\left(\rho,\eta\right)\Gamma\left(Z^\gamma\overset{*}{\tilde{\partial}}_\gamma+Y_a\overset{\cdot\ a}{\tilde{\partial}}\right)\left(\overset{*}{u}_x\right)
$$

are the real numbers

$$
\left(Y_{b'}-\left(\rho,\eta\right)\Gamma_{b'\gamma'}Z^{\gamma'}\right)\left(\overset{*}{u}_x\right).
$$

Since

$$
\begin{array}{l}
\left(Z^{\gamma'}\overset{*}{\tilde{\partial}}_{\gamma'}+Y_{b'}\overset{\cdot\ b'}{\tilde{\partial}}\right)\left(\overset{*}{u}_x\right)=Z^{\gamma'}\Lambda_{\gamma'}^{\gamma}\circ h\circ\overset{*}{\pi}\cdot\overset{*}{\tilde{\partial}}_\alpha\left(\overset{*}{u}_x\right)\\
\qquad+\left(Z^{\gamma'}\rho_{\gamma'}^{i'}\circ h\circ\overset{*}{\pi}\dfrac{\partial M_b^{a'}\circ\pi}{\partial x^{i'}}p_{a'}+M_b^{b'}Y_{b'}\right)\overset{\cdot\ b}{\tilde{\partial}}\left(\overset{*}{u}_x\right),
\end{array}
$$

it results that the components of

$$\overset{*}{\Pi} \circ (\rho,\eta)\,\Gamma\left(Z^{\gamma'}\overset{*}{\tilde{\partial}}_{\gamma'} + Y_{b'}\overset{\cdot\, b'}{\tilde{\partial}}\right)\left(\overset{*}{u}_x\right)$$

are the real numbers

$$\left(Z^{\acute{\gamma}}\rho^{\acute{k}}_{\acute{\gamma}} \circ h \circ \overset{*}{\pi}\tfrac{\partial M^{\acute{a}}_{b} \circ \overset{*}{\pi}}{\partial x^{\acute{k}}} p_{\acute{a}} + M^{\acute{b}}_{b} \circ \overset{*}{\pi} Y_{\acute{b}} - (\rho,\eta)\,\Gamma_{b\gamma} Z^{\acute{\gamma}} \Lambda^{\gamma}_{\acute{\gamma}} \circ h \circ \overset{*}{\pi}\right) M^{b}_{\acute{b}} \circ \overset{*}{\pi}\left(\overset{*}{u}_x\right),$$

where $\left\|M^{b}_{\acute{b}}\right\| = \left\|M^{\acute{b}}_{b}\right\|^{-1}$.

Therefore, we have:

$$\begin{aligned}&\left(Z^{\acute{\gamma}}\rho^{\acute{k}}_{\acute{\gamma}} \circ h \circ \overset{*}{\pi}\frac{\partial M^{\acute{a}}_{b} \circ \overset{*}{\pi}}{\partial x^{\acute{k}}} p_{\acute{a}} + M^{\acute{b}}_{b} \circ \overset{*}{\pi} Y_{\acute{b}} - (\rho,\eta)\,\Gamma_{b\gamma} Z^{\acute{\gamma}} \Lambda^{\gamma}_{\acute{\gamma}} \circ h \circ \overset{*}{\pi}\right) M^{b}_{\acute{b}} \circ \overset{*}{\pi}\\ &\qquad = Y_{\acute{b}} - (\rho,\eta)\,\Gamma_{\acute{b}\acute{\gamma}} Z^{\acute{\gamma}}.\end{aligned}$$

After some calculations we obtain:

$$(\rho,\eta)\,\Gamma_{\acute{b}\acute{\gamma}} = M^{b}_{\acute{b}} \circ \overset{*}{\pi}\left(-\rho^{k}_{\gamma} \circ h \circ \overset{*}{\pi} \cdot \tfrac{\partial M^{\acute{a}}_{b} \circ \overset{*}{\pi}}{\partial x^{k}} p_{\acute{a}} + (\rho,\eta)\,\Gamma_{b\gamma}\right) \Lambda^{\gamma}_{\acute{\gamma}} \circ h \circ \overset{*}{\pi}. \qquad q.e.d.$$

Remark 4.1.1 If we have a set of real local functions $(\rho,\eta)\,\Gamma_{b\gamma}$ which satisfies the relations of passing (4.1.8), then we have a (ρ,η)-connection $(\rho,\eta)\,\Gamma$ for the dual vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$.

Example 4.1.1 If Γ is a Ehresmann connection for the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ on components Γ_{bk}, then the differentiable real local functions

$$(\rho,\eta)\,\Gamma_{b\gamma} = \left(\rho^{k}_{\gamma} \circ h \circ \overset{*}{\pi}\right)\Gamma_{bk}$$

are the components of a (ρ,η)-connection $(\rho,\eta)\,\Gamma$ for the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ which will be called the *(ρ,η)-connection associated to the connection* Γ.

Definition 4.1.3 If $(\rho,\eta)\,\Gamma$ is a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, then for any

$$z = z^{\gamma} t_{\gamma} \in \Gamma(F,\nu,N)$$

the application

$$\begin{array}{ccc}\Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right) & \xrightarrow{(\rho,\eta)D_z} & \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right)\\ \overset{*}{u} = u_a s^a & \longmapsto & (\rho,\eta)\,D_z\overset{*}{u}\end{array}$$

defined by

$$(\rho,\eta)\,D_z\overset{*}{u} = z^{\gamma} \circ h\left(\rho^{k}_{\gamma} \circ h\tfrac{\partial u_b}{\partial x^k} - (\rho,\eta)\,\Gamma_{b\gamma} \circ \overset{*}{u}\right) s^b, \tag{4.1.9}$$

will be called the *covariant (ρ,η)-derivative associated to (ρ,η)-connection $(\rho,\eta)\,\Gamma$ with respect to section z.*

In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, the relations (4.1.9) become

$$\rho D_z \overset{*}{u} = z^\gamma \left(\rho_\gamma^k \frac{\partial u_b}{\partial x^k} - \rho \Gamma_{b\gamma} \circ \overset{*}{u} \right) s^b. \tag{4.1.9$'$}$$

In the classical case, $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$, the relations (4.1.9$'$) become

$$D_X \omega = X^k \left(\frac{\partial \omega_j}{\partial x^k} - \Gamma_{jk} \circ \omega \right) dx^j. \tag{4.1.9$''$}$$

Definition 4.1.4 We will say that the (ρ, η)-*connection* $(\rho, \eta)\,\Gamma$ *is homogeneous* or *linear* if the local real functions $(\rho, \eta)\,\Gamma_{b\gamma}$ are homogeneous or linear on the fibre of vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$ respectively.

Remark 4.1.2 If $(\rho, \eta)\,\Gamma$ is a linear (ρ, η)-connection for the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$, then for each local vector $(m+r)$-chart $\left(U, \overset{*}{s}_U \right)$ and for each local vector $(n+p)$-chart (V, t_V) such that $U \cap h^{-1}(V) \neq \phi$, there exists the differentiable real functions $\rho \Gamma_{b\gamma}^a$ defined on $U \cap h^{-1}(V)$ such that

$$(\rho, \eta)\,\Gamma_{b\gamma} \circ \overset{*}{u} = (\rho, \eta)\,\Gamma_{b\gamma}^a \cdot u_a, \tag{4.1.10}$$

for any $\overset{*}{u} = u_a s^a \in \Gamma \left(\overset{*}{E}, \overset{*}{\pi}, M \right)$.

The differentiable real local functions $(\rho, \eta)\,\Gamma_{b\gamma}^a$ will be called the *Christoffel coefficients of linear* (ρ, η)-*connection* $(\rho, \eta)\,\Gamma$.

Theorem 4.1.2 *If* $(\rho, \eta)\,\Gamma$ *is a linear* (ρ, η)-*connection for the vector bundle* $\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$, *then its components satisfy the law of transformation*

$$(\rho, \eta)\,\Gamma_{b'\gamma'}^{a'} = M_{b'}^b \left[-\rho_\gamma^k \circ h \frac{\partial M_b^{a'}}{\partial x^k} + (\rho, \eta)\,\Gamma_{b\gamma}^a M_a^{a'} \right] \Lambda_{\gamma'}^\gamma \circ h. \tag{4.1.11}$$

In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, *the relations* (4.1.11) *become*

$$\rho \Gamma_{b'\gamma'}^{a'} = M_{b'}^b \left[-\rho_\gamma^k \frac{\partial M_b^{a'}}{\partial x^k} + \rho \Gamma_{b\gamma}^a M_a^{a'} \right] \Lambda_{\gamma'}^\gamma. \tag{4.1.11$'$}$$

In the classical case, $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$, *the relations* (4.1.11$'$) *become*

$$\Gamma_{j'k'}^{i'} = \frac{\partial x^j}{\partial x^{j'}} \left[-\frac{\partial}{\partial x^k} \left(\frac{\partial x^{i'}}{\partial x^j} \right) + \Gamma_{jk}^i \frac{\partial x^{i'}}{\partial x^i} \right] \frac{\partial x^k}{\partial x^{k'}}. \tag{4.1.11$''$}$$

Remark 4.1.3 Since

$$\frac{\partial M_b^{a'}}{\partial x^i} M_{b'}^b + \frac{\partial M_{b'}^b}{\partial x^i} M_b^{a'} = 0,$$

it results that the relations (4.1.11) are equivalent with the relations (4.10$'$).

Definition 4.1.5 If $(\rho, \eta)\,\Gamma$ is a linear (ρ, η)-connection for the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$, then for any

$$z = z^\gamma t_\gamma \in \Gamma(F, \nu, N)$$

the application

$$\begin{array}{ccc}\Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right) & \xrightarrow{(\rho,\eta)D_z} & \Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right)\\ \overset{*}{u}=u_a s^a & \longmapsto & (\rho,\eta)D_z\overset{*}{u}\end{array}$$

defined by

$$(\rho,\eta)D_z\overset{*}{u}=z^\gamma\circ h\left(\rho_\gamma^k\circ h\frac{\partial u_b}{\partial x^k}-(\rho,\eta)\Gamma_{b\gamma}^a\cdot u_a\right)s^b \tag{4.1.12}$$

will be called the *covariant* (ρ,η)*-derivative associated to linear* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$ *with respect to section* z.

In the particular case of Lie algebroids, $(\eta,h)=(Id_M,Id_M)$, the relations (4.1.12) become

$$\rho D_z\overset{*}{u}=z^\gamma\left(\rho_\gamma^k\frac{\partial u_b}{\partial x^k}-\rho\Gamma_{b\gamma}^a\cdot u_a\right)s^b \tag{4.1.12$'$}$$

In the classical case, $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, the relations (4.1.12$'$) become

$$D_X\omega=X^k\left(\frac{\partial\omega_j}{\partial x^k}-\Gamma_{jk}^i\cdot\omega_i\right)dx^j \tag{4.1.12$''$}$$

In the next we use the same notation $(\rho,\eta)\,\Gamma$ for the linear (ρ,η)-connection for the vector bundle (E,π,M) or for its dual $\left(\overset{*}{E},\overset{*}{\pi},M\right)$

Remark 4.1.4 If $(\rho,\eta)\,\Gamma$ is a linear (ρ,η)-connection for the vector bundle (E,π,M) or for its dual $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ then, the tensor fields algebra

$$\left(\mathcal{T}\left(E,\pi,M\right),+,\cdot,\otimes\right)$$

is endowed with the (ρ,η)-derivative

$$\begin{array}{ccc}\Gamma\left(F,\nu,N\right)\times\mathcal{T}\left(E,\pi,M\right) & \xrightarrow{(\rho,\eta)D} & \mathcal{T}\left(E,\pi,M\right)\\ (z,T) & \longmapsto & (\rho,\eta)\,D_zT\end{array}$$

defined for a tensor field $T\in\mathcal{T}_q^p\left(E,\pi,M\right)$ by the relation:

$$\begin{array}{c}(\rho,\eta)\,D_zT\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)=\Gamma\left(\rho,\eta\right)(z)\left(T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)\right)\\ -T\left((\rho,\eta)\,D_z\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)-...-T\left(\overset{*}{u}_1,...,(\rho,\eta)\,D_z\overset{*}{u}_p,u_1,...,u_q\right)\\ -T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,(\rho,\eta)\,D_zu_1,...,u_q\right)-...-T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,(\rho,\eta)\,D_zu_q\right).\end{array} \tag{4.1.13}$$

Moreover, it satisfies the condition

$$(\rho,\eta)\,D_{f_1z_1+f_2z_2}T=f_1\,(\rho,\eta)\,D_{z_1}T+f_2\,(\rho,\eta)\,D_{z_2}T. \tag{4.1.14}$$

Consequently, if the tensor algebra $\left(\mathcal{T}\left(E,\pi,M\right),+,\cdot,\otimes\right)$ is endowed with a (ρ,η)-derivative defined for a tensor field $T\in\mathcal{T}_q^p\left(E,\pi,M\right)$ by (4.1.13) which satisfies the condition (4.1.14), then we can endowed (E,π,M) with a linear (ρ,η)-connection $(\rho,\eta)\,\Gamma$ such that its components are defined by the equality:

$$(\rho,\eta)\,D_{t_\gamma}s_b=(\rho,\eta)\,\Gamma_{b\gamma}^a s_a$$

or

$$(\rho,\eta)\, D_{t_\gamma} s^a = -\,(\rho,\eta)\,\Gamma^a_{b\gamma} s^b.$$

The (ρ,η)-derivative defined by (4.1.13) will be called the *covariant* (ρ,η)*-derivative.*

After some calculations, we obtain:

$$\begin{array}{ll} & (\rho,\eta)\, D_z \left(T^{a_1,\dots,a_p}_{b_1,\dots,b_q} s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}\right)\\ & = z^\gamma\circ h\left(\rho^k_\gamma\circ h\dfrac{\partial T^{a_1,\dots,a_p}_{b_1,\dots,b_q}}{\partial x^k}+(\rho,\eta)\,\Gamma^{a_1}_{a\gamma}T^{a,a_2\dots,a_p}_{b_1,\dots,b_q}\right.\\ (4.1.15) & +(\rho,\eta)\,\Gamma^{a_2}_{a\gamma}T^{a_1,a\dots,a_p}_{b_1,\dots,b_q}+\dots+(\rho,\eta)\,\Gamma^{a_p}_{a\gamma}T^{a_1,a_2\dots,a}_{b_1,\dots,b_q}-\dots\\ & -(\rho,\eta)\,\Gamma^{b}_{b_1\gamma}T^{a_1,a_2\dots,a_p}_{b,b_2,\dots,b_q}-(\rho,\eta)\,\Gamma^{b}_{b_2\gamma}T^{a_1,a_2\dots,a_p}_{b_1,b,\dots,b_q}-\dots\\ & \left.-(\rho,\eta)\,\Gamma^{b}_{b_q\gamma}T^{a_1,a_2\dots,a_p}_{b_1,b_2,\dots,b}\right) s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}\\ & \overset{put}{=} z^\gamma\circ h T^{a_1,\dots,a_p}_{b_1,\dots,b_q|\gamma} s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}.\end{array}$$

We remark that if $(\rho,\eta)\,\Gamma$ is the linear (ρ,η)-connection associated to the Ehresmann linear connection Γ, namely $(\rho,\eta)\,\Gamma^a_{b\alpha}=\left(\rho^k_\alpha\circ h\right)\Gamma^a_{bk}$, then

$$T^{a_1,\dots,a_p}_{b_1,\dots,b_q|\gamma}=\left(\rho^k_\gamma\circ h\right)T^{a_1,\dots,a_p}_{b_1,\dots,b_q|k}.$$

In the particular case of Lie algebroids, $(\eta,h)=(Id_M,Id_M)$, the relations (4.1.15) become

$$\begin{array}{ll} & \rho D_z \left(T^{a_1,\dots,a_p}_{b_1,\dots,b_q} s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}\right)\\ & = z^\gamma\left(\rho^k_\gamma\dfrac{\partial T^{a_1,\dots,a_p}_{b_1,\dots,b_q}}{\partial x^k}+\rho\Gamma^{a_1}_{a\gamma}T^{a,a_2\dots,a_p}_{b_1,\dots,b_q}\right.\\ (4.1.15') & +\rho\Gamma^{a_2}_{a\gamma}T^{a_1,a\dots,a_p}_{b_1,\dots,b_q}+\dots+\rho\Gamma^{a_p}_{a\gamma}T^{a_1,a_2\dots,a}_{b_1,\dots,b_q}-\dots\\ & -\rho\Gamma^{b}_{b_1\gamma}T^{a_1,a_2\dots,a_p}_{b,b_2,\dots,b_q}-\rho\Gamma^{b}_{b_2\gamma}T^{a_1,a_2\dots,a_p}_{b_1,b,\dots,b_q}-\dots\\ & \left.-\rho\Gamma^{b}_{b_q\gamma}T^{a_1,a_2\dots,a_p}_{b_1,b_2,\dots,b}\right) s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}\\ & \overset{put}{=} z^\gamma T^{a_1,\dots,a_p}_{b_1,\dots,b_q|\gamma} s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}.\end{array}$$

In the classical case, $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, the relations (4.1.15′) become

$$\begin{array}{ll} & D_X\left(T^{i_1\dots i_p}_{j_1\dots j_q}\partial_{i_1}\otimes\dots\otimes\partial_{i_p}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_q}\right)\\ & = X^k\left(\frac{\partial T^{i_1\dots i_p}_{j_1\dots j_q}}{\partial x^k}+\Gamma^{i_1}_{ik}T^{ii_2\dots i_p}_{j_1\dots j_q}+\dots+\Gamma^{i_p}_{ik}T^{i_1\dots i_{p-1}i}_{j_1\dots j_q}\right.\\ (4.1.15'') & \left.-\Gamma^{j}_{j_1k}T^{i_1\dots i_p}_{jj_2\dots j_q}-\dots-\Gamma^{j}_{j_qk}T^{i_1\dots i_p}_{j_1\dots j_{q-1}j}\right)\partial_{i_1}\otimes\dots\otimes\partial_{i_p}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_q}\\ & \overset{put}{=} X^k T^{i_1\dots i_p}_{j_1\dots j_q|k}\partial_{i_1}\otimes\dots\otimes\partial_{i_p}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_q}.\end{array}$$

5 Torsion and curvature. Formulas of Ricci and Bianchi type

We apply our theory for the diagram:

$$(5.1)\qquad \begin{array}{ccc} E & & \left(F, [,]_{F,h}, (\rho, Id_N)\right) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array},$$

where $(E, \pi, M) \in |\mathbf{B^v}|$ and $\left((F, \nu, N), [,]_{F,h}, (\rho, Id_N)\right) \in |\mathbf{GLA}|$.

Let $\rho\Gamma$ be a linear ρ-connection for the vector bundle (E, π, M) by components $\rho\Gamma^a_{b\alpha}$.

Using the components of the linear ρ-connection $\rho\Gamma$, then we obtain a linear ρ-connection $\rho\dot{\Gamma}$ for the vector bundle (E, π, M) given by the diagram:

$$(5.2)\qquad \begin{array}{ccc} E & & \left(h^*F, [,]_{h^*F}, \left(\overset{h^*F}{\rho}, Id_M\right)\right) \\ \pi \downarrow & & \downarrow h^*\nu \\ M & \xrightarrow{Id_M} & M \end{array}$$

If $(E, \pi, M) = (F, \nu, N)$, then, using the components of the linear ρ-connection $\rho\Gamma$, we can consider a linear ρ-connection $\rho\ddot{\Gamma}$ for the vector bundle $(h^*E, h^*\pi, M)$ given by the diagram:

$$(5.3)\qquad \begin{array}{ccc} h^*E & & \left(h^*E, [,]_{h^*E}, \left(\overset{h^*E}{\rho}, Id_M\right)\right) \\ h^*\pi \downarrow & & \downarrow h^*\pi \\ M & \xrightarrow{Id_M} & M \end{array}$$

Definition 5.1 If $(E, \pi, M) = (F, \nu, N)$, then the application

$$\begin{array}{ccc} \Gamma\left(h^*E, h^*\pi, M\right)^2 & \xrightarrow{(\rho,h)\mathbb{T}} & \Gamma\left(h^*E, h^*\pi, M\right) \\ (U, V) & \longrightarrow & \rho\mathbb{T}\left(U, V\right) \end{array}$$

defined by:

$$(5.4)\qquad (\rho, h)\,\mathbb{T}\left(U, V\right) = \rho\ddot{D}_U V - \rho\ddot{D}_V U - [U, V]_{h^*E},$$

for any $U, V \in \Gamma\left(h^*E, h^*\pi, M\right)$, will be called *$(\rho, h)$-torsion associated to the linear ρ-connection* $\rho\Gamma$.

In the particular case of Lie algebroids, $h = Id_M$, we obtain the application

$$\begin{array}{ccc} \Gamma\left(E, \pi, M\right)^2 & \xrightarrow{\rho\mathbb{T}} & \Gamma\left(E, \pi, M\right) \\ (u, v) & \longrightarrow & \rho\mathbb{T}\left(u, v\right) \end{array}$$

defined by:

$$(5.4')\qquad \rho\mathbb{T}\left(u, v\right) = \rho D_u v - \rho D_v u - [u, v]_E,$$

for any $u, v \in \Gamma\left(E, \pi, M\right)$, which will be called the *ρ-torsion associated to the linear ρ-connection* $\rho\Gamma$.

In the classical case, $\rho = Id_{TM}$, we obtain the torsion $\mathbb{T}$ associated to the linear connection Γ.

Proposition 5.1 *The (ρ, h)-torsion $(\rho, h)\,\mathbb{T}$ associated to the linear ρ-connection $\rho\Gamma$ is $\mathbb{R}$-bilinear and antisymmetric.*

If

$$(\rho, h)\,\mathbb{T}\,(S_a, S_b) \overset{put}{=} (\rho, h)\,\mathbb{T}^c{}_{ab} S_c$$

then

$$(\rho, h)\,\mathbb{T}^c{}_{ab} = \rho\Gamma^c_{ab} - \rho\Gamma^c_{ba} - L^c_{ab} \circ h. \tag{5.5}$$

In the particular case of Lie algebroids, $h = Id_M$, we have $\rho\mathbb{T}\,(s_a, s_b) \overset{put}{=} \rho\mathbb{T}^c_{ab} s_c$ and

$$\rho\mathbb{T}^c{}_{ab} = \rho\Gamma^c_{ab} - \rho\Gamma^c_{ba} - L^c_{ab}. \tag{5.5'}$$

In the classical case, $(\rho, h) = (Id_{TM}, Id_M)$, the equality $(5.5')$ becomes:

$$\mathbb{T}^i{}_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}. \tag{5.5''}$$

Definition 5.2 The application

$$\begin{array}{ccc} (\Gamma\,(h^*F, h^*\nu, M)^2 \times \Gamma(E, \pi, M) & \xrightarrow{(\rho,h)\mathbb{R}} & \Gamma(E, \pi, M) \\ ((Z, V), u) & \longrightarrow & (\rho, h)\,\mathbb{R}(Z, V)u \end{array}$$

defined by

$$(\rho, h)\,\mathbb{R}\,(Z, V)\,u = \rho\dot{D}_Z\left(\rho\dot{D}_V u\right) - \rho\dot{D}_V\left(\rho\dot{D}_Z u\right) - \rho\dot{D}_{[Z,V]_{h^*F}} u, \tag{5.6}$$

for any $Z, V \in \Gamma\,(h^*F, h^*\nu, M)$ and $u \in \Gamma\,(E, \pi, M)$, will be called (ρ, h)-*curvature associated to the linear ρ-connection* $\rho\Gamma$.

In the particular case of Lie algebroids, $h = Id_M$, we obtain the application

$$\begin{array}{ccc} \Gamma\,(F, \nu, M)^2 \times \Gamma(E, \pi, M) & \xrightarrow{\rho\mathbb{R}} & \Gamma(E, \pi, M) \\ ((z, v), u) & \longrightarrow & \rho\mathbb{R}(z, v)u \end{array}$$

defined by

$$\rho\mathbb{R}\,(z, v)\,u = \rho D_z\,(\rho D_v u) - \rho D_v\,(\rho D_z u) - \rho D_{[z,v]_F} u, \tag{5.6'}$$

for any $z, v \in \Gamma\,(F, \nu, M)$ and $u \in \Gamma\,(E, \pi, M)$, which will be called ρ-*curvature associated to the linear ρ-connection* $\rho\Gamma$.

In the classical case, $(\rho, h) = (Id_{TM}, Id_M)$, we obtain the curvature $\mathbb{R}$ associated to the linear connection Γ.

Proposition 5.2 *The (ρ, h)-curvature $(\rho, h)\,\mathbb{R}$ associated to the linear ρ-connection $\rho\Gamma$, is $\mathbb{R}$-linear in each argument and antisymmetric in the first two arguments.*

If

$$(\rho, h)\,\mathbb{R}\,(T_\beta, T_\alpha)\,s_b \overset{put}{=} (\rho, h)\,\mathbb{R}^a_{b\ \alpha\beta} s_a,$$

then

$$\begin{array}{rl} (\rho, h)\,\mathbb{R}^a_{b\ \alpha\beta} & = \rho^j_\beta \circ h\frac{\partial\rho\Gamma^a_{b\alpha}}{\partial x^j} + \rho\Gamma^a_{e\beta}\rho\Gamma^e_{b\alpha} - \rho^i_\alpha \circ h\frac{\partial\rho\Gamma^a_{b\beta}}{\partial x^i} \\ & -\rho\Gamma^a_{e\alpha}\rho\Gamma^e_{b\beta} + \rho\Gamma^a_{b\gamma}L^\gamma_{\alpha\beta} \circ h. \end{array} \tag{5.7}$$

In the particular case of Lie algebroids, $h=Id_M$*, we obtain* $\rho\mathbb{R}\left(t_\beta,t_\alpha\right)s_b\overset{put}{=}\rho\mathbb{R}_{b\ \alpha\beta}^{a}s_a$*, and*

$$\rho\mathbb{R}_{b\ \alpha\beta}^{a}=\rho_\beta^j\frac{\partial\rho\Gamma_{b\alpha}^{a}}{\partial x^j}+\rho\Gamma_{e\beta}^{a}\rho\Gamma_{b\alpha}^{e}-\rho_\alpha^i\frac{\partial\rho\Gamma_{b\beta}^{a}}{\partial x^i}-\rho\Gamma_{e\alpha}^{a}\rho\Gamma_{b\beta}^{e}+\rho\Gamma_{b\gamma}^{a}L_{\alpha\beta}^{\gamma}. \tag{5.7'}$$

In the classical case, $(\rho,h)=\left(Id_{TM},Id_M\right)$*, we obtain* $\mathbb{R}\left(\partial_k,\partial_h\right)s_b\overset{put}{=}\mathbb{R}_{b\ hk}^{a}s_a$*, and the equality* $(5.7')$ *becomes:*

$$\mathbb{R}_{b\ hk}^{a}=\frac{\partial\Gamma_{bh}^{a}}{\partial x^k}+\Gamma_{ek}^{a}\Gamma_{bh}^{e}-\frac{\partial\Gamma_{bk}^{a}}{\partial x^h}-\Gamma_{eh}^{a}\Gamma_{bk}^{e}. \tag{5.7''}$$

Theorem 5.1 *For any* $u^as_a\in\Gamma\left(E,\pi,M\right)$ *we shall use the notation*

$$u^{a}{}_{|\alpha\beta}=\rho_\beta^j\circ h\frac{\partial}{\partial x^j}\left(u^{a_1}{}_{|\alpha}\right)+\rho\Gamma_{b_\beta}^{a_1}u^{b}{}_{|\alpha}, \tag{5.8}$$

and we verify the formulas:

$$u^{a_1}{}_{|\alpha\beta}-u^{a_1}{}_{|\beta\alpha}=u^a\left(\rho,h\right)\mathbb{R}_{a\ \alpha\beta}^{a_1}-u^{a_1}{}_{|\gamma}L_{\alpha\beta}^{\gamma}\circ h. \tag{5.9}$$

After some calculations, we obtain

$$\left(\rho,h\right)\mathbb{R}_{a\ \alpha\beta}^{a_1}=u_a\left(u^{a_1}{}_{|\alpha\beta}-u^{a_1}{}_{|\beta\alpha}+u^{a_1}{}_{|\gamma}L_{\alpha\beta}^{\gamma}\circ h\right), \tag{5.10}$$

where $u_as^a\in\Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right)$ *such that* $u_au^b=\delta_a^b$.

In the particular case of Lie algebroids, $h=Id_M$*, the relations* (5.10) *become*

$$\rho\mathbb{R}_{a\ \alpha\beta}^{a_1}=u_a\left(u^{a_1}{}_{|\alpha\beta}-u^{a_1}{}_{|\beta\alpha}+u^{a_1}{}_{|\gamma}L_{\alpha\beta}^{\gamma}\right). \tag{5.10'}$$

In the classical case, $(\rho,h)=\left(Id_{TM},Id_M\right)$*, the relations* $(5.10')$ *become*

$$\mathbb{R}_{a\ ij}^{a_1}=u_a\left(u^{a_1}{}_{|ij}-u^{a_1}{}_{|ji}\right). \tag{5.10''}$$

Proof. Since

$$\begin{aligned}
u^{a_1}{}_{|\alpha\beta}&=\rho_\beta^j\circ h\left(\frac{\partial}{\partial x^j}\left(\rho_\alpha^i\circ h\frac{\partial u^{a_1}}{\partial x^i}+\rho\Gamma_{a\alpha}^{a_1}u^a\right)\right)\\
&+\rho\Gamma_{b\beta}^{a_1}\left(\rho_\alpha^i\circ h\frac{\partial u^b}{\partial x^i}+\rho\Gamma_{a\alpha}^{b}u^a\right)\\
&=\rho_\beta^j\circ h\frac{\partial\rho_\alpha^i\circ h}{\partial x^j}\frac{\partial u^{a_1}}{\partial x^i}+\rho_\beta^j\circ h\rho_\alpha^i\circ h\frac{\partial}{\partial x^j}\left(\frac{\partial u^{a_1}}{\partial x^i}\right)\\
&+\rho_\beta^j\circ h\frac{\partial\rho\Gamma_{a\alpha}^{a_1}}{\partial x^j}u^a+\rho_\beta^j\circ h\rho\Gamma_{a\alpha}^{a_1}\frac{\partial u^a}{\partial x^j}\\
&+\rho_\alpha^i\circ h\rho\Gamma_{b\beta}^{a_1}\frac{\partial u^b}{\partial x^i}+\rho\Gamma_{b\beta}^{a_1}\rho\Gamma_{a\alpha}^{b}u^a
\end{aligned}$$

and

$$
\begin{aligned}
u^{a_1}{}_{|\beta\alpha} &= \rho^i_\alpha \circ h \left(\frac{\partial}{\partial x^i} \left(\rho^j_\beta \circ h \frac{\partial u^{a_1}}{\partial x^j} + \rho \Gamma^{a_1}_{a\beta} u^a \right) \right) \\
&+ \rho \Gamma^{a_1}_{b\alpha} \left(\rho^j_\beta \circ h \frac{\partial u^b}{\partial x^j} + \rho \Gamma^b_{a\beta} u^a \right) \\
&= \rho^i_\alpha \circ h \frac{\partial \rho^j_\beta \circ h}{\partial x^i} \frac{\partial u^{a_1}}{\partial x^j} + \rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial}{\partial x^i} \left(\frac{\partial u^{a_1}}{\partial x^j} \right) \\
&+ \rho^i_\alpha \circ h \frac{\partial \rho \Gamma^{a_1}_{a\beta}}{\partial x^i} u^a + \rho^i_\alpha \circ h \rho \Gamma^{a_1}_{a\beta} \frac{\partial u^a}{\partial x^i} \\
&+ \rho^j_\beta \circ h \rho \Gamma^{a_1}_{b\alpha} \frac{\partial u^b}{\partial x^j} + \rho \Gamma^{a_1}_{b\alpha} \rho \Gamma^b_{a\beta} u^a,
\end{aligned}
$$

it results that

$$
\begin{aligned}
u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} &= \rho^j_\beta \circ h \frac{\partial \rho^i_\alpha \circ h}{\partial x^j} \frac{\partial u^{a_1}}{\partial x^i} - \rho^i_\alpha \circ h \frac{\partial \rho^j_\beta \circ h}{\partial x^i} \frac{\partial u^{a_1}}{\partial x^j} \\
&+ \left(\rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial^2 u^{a_1}}{\partial x^i \partial x^j} - \rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial^2 u^{a_1}}{\partial x^j \partial x^i} \right) \\
&+ \left(\rho^j_\beta \circ h \frac{\partial \rho \Gamma^{a_1}_{a\alpha}}{\partial x^j} u^a - \rho^i_\alpha \circ h \frac{\partial \rho \Gamma^{a_1}_{a\beta}}{\partial x^i} u^a \right) \\
&+ \left(\rho^j_\beta \circ h \rho \Gamma^{a_1}_{a\alpha} \frac{\partial u^a}{\partial x^j} - \rho^j_\beta \circ h \rho \Gamma^{a_1}_{b\alpha} \frac{\partial u^b}{\partial x^j} \right) \\
&+ \left(\rho^i_\alpha \circ h \rho \Gamma^{a_1}_{b\beta} \frac{\partial u^b}{\partial x^i} - \rho^i_\alpha \circ h \rho \Gamma^{a_1}_{a\beta} \frac{\partial u^a}{\partial x^i} \right) \\
&+ \rho \Gamma^{a_1}_{b\beta} \rho \Gamma^b_{a\alpha} u^a - \rho \Gamma^{a_1}_{b\alpha} \rho \Gamma^b_{a\beta} u^a .
\end{aligned}
$$

After some calculations, we obtain:

$$
\begin{aligned}
u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} &= L^\gamma_{\beta\alpha} \circ h \rho^k_\gamma \circ h \frac{\partial u^{a_1}}{\partial x^k} \\
&+ \left(\rho^j_\beta \circ h \frac{\partial \rho \Gamma^{a_1}_{a\alpha}}{\partial x^j} u^a - \rho^i_\alpha \circ h \frac{\partial \rho \Gamma^{a_1}_{a\beta}}{\partial x^i} u^a \right) \\
&+ \rho \Gamma^{a_1}_{b\beta} \rho \Gamma^b_{a\alpha} u^a - \rho \Gamma^{a_1}_{b\alpha} \rho \Gamma^b_{a\beta} u^a .
\end{aligned}
$$

Since

$$
\begin{array}{rcl}
u^a (\rho, h) \mathbb{R}^{a_1}_{a\ \alpha\beta} & = & u^a \left(\rho^j_\beta \circ h \dfrac{\partial \rho \Gamma^{a_1}_{a\alpha}}{\partial x^j} + \rho \Gamma^{a_1}_{e\beta} \rho \Gamma^e_{a\alpha} - \rho^i_\alpha \circ h \dfrac{\partial \rho \Gamma^{a_1}_{a\beta}}{\partial x^i} \right. \\
& & \left. - \rho \Gamma^{a_1}_{e\alpha} \rho \Gamma^e_{a\beta} - \rho \Gamma^{a_1}_{a\gamma} L^\gamma_{\beta\alpha} \circ h \right) .
\end{array}
$$

and

$$
u^{a_1}{}_{|\gamma} L^\gamma_{\alpha\beta} \circ h = \left(\rho^k_\gamma \circ h \frac{\partial u^{a_1}}{\partial x^k} + \rho \Gamma^{a_1}_{a\gamma} u^a \right) L^\gamma_{\alpha\beta} \circ h
$$

it results that

$$u^{a}\left(\rho,h\right)\mathbb{R}_{a\ \alpha\beta}^{a_1}-u^{a_1}_{\ |\gamma}L_{\alpha\beta}^{\gamma}\circ h=-L_{\alpha\beta}^{\gamma}\circ h\rho_{\gamma}^{k}\circ h\frac{\partial u^{a_1}}{\partial x^{k}}$$
$$+\left(\rho_{\beta}^{j}\circ h\frac{\partial\rho\Gamma_{a\alpha}^{a_1}}{\partial x^{j}}u^{a}-\rho_{\alpha}^{i}\circ h\frac{\partial\rho\Gamma_{a\beta}^{a_1}}{\partial x^{i}}u^{a}\right)$$
$$+\rho\Gamma_{b\beta}^{a_1}\rho\Gamma_{a\alpha}^{b}u^{a}-\rho\Gamma_{b\alpha}^{a_1}\rho\Gamma_{a\beta}^{b}u^{a}.$$

q.e.d.

Lemma 5.1 *If* $(E,\pi,M)=(F,\nu,N)$, *then, for any* $u^{a}s_{a}\in\Gamma\left(E,\pi,M\right)$, *we have that* $u^{a}_{\ |c}$, $a,c\in\overline{1,n}$ are the *components of a tensor field of* $(1,1)$ *type.*

Proof. Let U and U' be two vector local $(m+n)$-charts such that $U\cap U'\neq\phi$.

Since $u^{a'}\left(x\right)=M_{a}^{a'}\left(x\right)u^{a}\left(x\right)$, for any $x\in U\cap U'$, it results that

$$\rho_{c'}^{k'}\circ h\left(x\right)\tfrac{\partial u^{a'}(x)}{\partial x^{k'}}=\rho_{c'}^{k'}\circ h\left(x\right)\tfrac{\partial}{\partial x^{k'}}\left(M_{a}^{a'}\left(x\right)\right)u^{a}\left(x\right)+M_{a}^{a'}\left(x\right)\rho_{c'}^{k'}\circ h\left(x\right)\tfrac{\partial u^{a}(x)}{\partial x^{k'}}.\quad(1)$$

Since , for any $x\in U\cap U'$, we have

$$\rho\Gamma_{b'c'}^{a'}(x)=M_{a}^{a'}(x)\left(\rho_{c}^{k}\circ h(x)\tfrac{\partial}{\partial x^{k}}(M_{b'}^{a}(x))+\rho\Gamma_{bc}^{a}(x)M_{b'}^{b}(x)\right)M_{c'}^{c}(x),\quad(2)$$

and

$$0=\tfrac{\partial}{\partial x^{k'}}\left(M_{a}^{a'}\left(x\right)M_{b'}^{a}\left(x\right)\right)=\tfrac{\partial}{\partial x^{k'}}\left(M_{a}^{a'}\left(x\right)\right)M_{b'}^{a}\left(x\right)+M_{a}^{a'}\left(x\right)\tfrac{\partial}{\partial x^{k'}}\left(M_{b'}^{a}\left(x\right)\right)\quad(3)$$

it results that

$$\begin{aligned}\rho\Gamma_{b'c'}^{a'}\left(x\right)u^{b'}\left(x\right)=&-\rho_{c'}^{k'}\circ h\left(x\right)\frac{\partial}{\partial x^{k'}}\left(M_{a}^{a'}\left(x\right)\right)u^{a}\left(x\right)\\&+M_{a}^{a'}\left(x\right)\rho\Gamma_{bc}^{a}\left(x\right)u^{b}\left(x\right)M_{c'}^{c}\left(x\right).\end{aligned}\quad(4)$$

Summing the equalities (1) and (4), it results the conclusion of lemma. *q.e.d.*

Theorem 5.2 *If* $(E,\pi,M)=(F,\nu,N)$, *then, for any*

$$u^{a}s_{a}\in\Gamma\left(E,\pi,M\right),$$

we shall use the notation

$$u^{a_1}_{\ |a|b}=u^{a_1}_{\ |ab}-\rho\Gamma_{ab}^{d}u^{a_1}_{\ |d}\qquad(5.11)$$

and we verify the formulas of Ricci type

$$u^{a_1}_{\ |a|b}-u^{a_1}_{\ |b|a}+\left(\rho,h\right)\mathbb{T}^{d}_{\ ab}u^{a_1}_{\ |d}=u^{d}\left(\rho,h\right)\mathbb{R}_{d\ ab}^{a_1}-u^{a_1}_{\ |c}L_{ab}^{c}\circ h\qquad(5.12)$$

In the particular case of Lie algebroids, $h=Id_{M}$, *the relations* (5.12) *become*

$$u^{a_1}_{\ |a|b}-u^{a_1}_{\ |b|a}+\rho\mathbb{T}^{d}_{\ ab}u^{a_1}_{\ |d}=u^{d}\rho\mathbb{R}_{d\ ab}^{a_1}-u^{a_1}_{\ |c}L_{ab}^{c}\qquad(5.12')$$

In the classical case,$(\rho,h)=\left(Id_{TM},Id_{M}\right)$, *the relations* $(5.12')$ *become*

$$u^{i_1}_{\ |i|j}-u^{i_1}_{\ |i|j}+\mathbb{T}^{k}_{\ ij}u^{i_1}_{\ |k}=u^{k}\mathbb{R}_{k\ ij}^{i_1}\qquad(5.12'')$$

Theorem 5.3 *For any $u_a s^a \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ we shall use the notation*

$$u_{b_1|\alpha\beta} = \rho_\beta^j \circ h\tfrac{\partial}{\partial x^j}\left(u_{b_1|\alpha}\right) - \rho\Gamma_{b_1\beta}^b u_{b|\alpha} \tag{5.13}$$

and we verify the formulas:

$$u_{b_1|\alpha\beta} - u_{b_1|\beta\alpha} = -u_b\left(\rho, h\right)\mathbb{R}_{b_1\ \alpha\beta}^b - u_{b_1|\gamma}L_{\alpha\beta}^\gamma \circ h \tag{5.14}$$

After some calculations, we obtain

$$\left(\rho, h\right)\mathbb{R}_{b_1\ \alpha\beta}^b = u^b\left(-u_{b_1|\alpha\beta} + u_{b_1|\beta\alpha} - u_{b_1|\gamma}L_{\alpha\beta}^\gamma \circ h\right), \tag{5.15}$$

where $u^a s_a \in \Gamma\left(E, \pi, M\right)$ such that $u_a u^b = \delta_a^b$.

In the particular case of Lie algebroids, $h = Id_M$, the relations (5.15) *become*

$$\rho\mathbb{R}_{b_1\ \alpha\beta}^b = u^b\left(-u_{b_1|\alpha\beta} + u_{b_1|\beta\alpha} - u_{b_1|\gamma}L_{\alpha\beta}^\gamma\right). \tag{5.15$'$}$$

In the classical case, $\left(\rho, h\right) = \left(Id_{TM}, Id_M\right)$, the relations (5.15′) *become*

$$\mathbb{R}_{b_1\ ij}^b = u^b\left(-u_{b_1|ij} + u_{b_1|ji}\right). \tag{5.15$''$}$$

Proof. Since

$$\begin{aligned}
u_{b_1|\alpha\beta} &= \rho_\beta^j \circ h\left(\frac{\partial}{\partial x^j}\left(\rho_\alpha^i \circ h\frac{\partial u_{b_1}}{\partial x^i} - \rho\Gamma_{b_1\alpha}^b u_b\right)\right)\\
&- \rho\Gamma_{b_1\beta}^b\left(\rho_\alpha^i \circ h\frac{\partial u_b}{\partial x^i} - \rho\Gamma_{b\alpha}^a u_a\right)\\
&= \rho_\beta^j \circ h\frac{\partial\rho_\alpha^i \circ h}{\partial x^j}\frac{\partial u_{b_1}}{\partial x^i} + \rho_\beta^j \circ h\rho_\alpha^i \circ h\frac{\partial}{\partial x^j}\left(\frac{\partial u_{b_1}}{\partial x^i}\right)\\
&- \rho_\beta^j \circ h\frac{\partial\rho\Gamma_{b_1\alpha}^b}{\partial x^j}u_b - \rho_\beta^j \circ h\rho\Gamma_{b_1\alpha}^b\frac{\partial u_b}{\partial x^j}\\
&- \rho_\alpha^i \circ h\rho\Gamma_{b_1\beta}^b\frac{\partial u_b}{\partial x^i} + \rho\Gamma_{b_1\beta}^b\rho\Gamma_{b\alpha}^a u_a
\end{aligned}$$

and

$$\begin{aligned}
u_{b_1|\beta\alpha} &= \rho_\alpha^i \circ h\left(\frac{\partial}{\partial x^i}\left(\rho_\beta^j \circ h\frac{\partial u_{b_1}}{\partial x^j} - \rho\Gamma_{b_1\beta}^b u_b\right)\right)\\
&- \rho\Gamma_{b_1\alpha}^b\left(\rho_\beta^j \circ h\frac{\partial u_b}{\partial x^j} - \rho\Gamma_{b\beta}^a u_a\right)\\
&= \rho_\alpha^i \circ h\frac{\partial\rho_\beta^j \circ h}{\partial x^i}\frac{\partial u_{b_1}}{\partial x^i} + \rho_\beta^j \circ h\rho_\alpha^i \circ h\frac{\partial}{\partial x^i}\left(\frac{\partial u_{b_1}}{\partial x^j}\right)\\
&- \rho_\alpha^i \circ h\frac{\partial\rho\Gamma_{b_1\beta}^b}{\partial x^i}u_b - \rho_\alpha^i \circ h\rho\Gamma_{b_1\beta}^b\frac{\partial u_b}{\partial x^i}\\
&- \rho_\beta^j \circ h\rho\Gamma_{b_1\alpha}^b\frac{\partial u_b}{\partial x^i} + \rho\Gamma_{b_1\alpha}^b\rho\Gamma_{b\beta}^a u_a
\end{aligned}$$

it results that

$$
\begin{aligned}
u_{b_1|\alpha\beta} - u_{b_1|\beta\alpha} &= \rho_\beta^j \circ h \frac{\partial \rho_\alpha^i \circ h}{\partial x^j} \frac{\partial u_{b_1}}{\partial x^i} - \rho_\alpha^i \circ h \frac{\partial \rho_\beta^j \circ h}{\partial x^i} \frac{\partial u_{b_1}}{\partial x^j} \\
&+ \rho_\beta^j \circ h \rho_\alpha^i \circ h \frac{\partial}{\partial x^j} \left(\frac{\partial u_{b_1}}{\partial x^i} \right) - \rho_\beta^j \circ h \rho_\alpha^i \circ h \frac{\partial}{\partial x^i} \left(\frac{\partial u_{b_1}}{\partial x^j} \right) \\
&+ \rho_\alpha^i \circ h \frac{\partial \rho \Gamma_{b_1\beta}^b}{\partial x^i} u_b - \rho_\beta^j \circ h \frac{\partial \rho \Gamma_{b_1\alpha}^b}{\partial x^j} u_b \\
&+ \rho_\beta^j \circ h \rho \Gamma_{b_1\alpha}^b \frac{\partial u_b}{\partial x^j} - \rho_\beta^j \circ h \rho \Gamma_{b_1\alpha}^b \frac{\partial u_b}{\partial x^j} \\
&+ \rho_\alpha^i \circ h \rho \Gamma_{b_1\alpha}^b \frac{\partial u_b}{\partial x^i} - \rho_\alpha^i \circ h \rho \Gamma_{b_1\alpha}^b \frac{\partial u_b}{\partial x^i} \\
&+ \rho \Gamma_{b_1\beta}^b \rho \Gamma_{b\alpha}^a u_a - \rho \Gamma_{b_1\alpha}^b \rho \Gamma_{b\beta}^a u_a.
\end{aligned}
$$

After some calculations, we obtain:

$$
\begin{aligned}
u_{b_1|\alpha\beta} - u_{b_1|\beta\alpha} \quad &= L_{\beta\alpha}^\gamma \circ h \rho_\gamma^k \circ h \tfrac{\partial u_{b_1}}{\partial x^k} \\
&+ \left(\rho_\alpha^i \circ h \tfrac{\partial \rho \Gamma_{b_1\beta}^b}{\partial x^i} u_b - \rho_\beta^j \circ h \tfrac{\partial \rho \Gamma_{b_1\alpha}^b}{\partial x^j} u_b \right) \\
&+ \rho \Gamma_{b_1\beta}^b \rho \Gamma_{b\alpha}^a u_a - \rho \Gamma_{b_1\alpha}^b \rho \Gamma_{b\beta}^a u_a.
\end{aligned}
$$

Since

$$
\begin{aligned}
u_b\left(\rho, h\right) \mathbb{R}_{b_1\alpha\beta}^b = u_b \Bigg(& \rho_\beta^j \circ h \frac{\partial \rho \Gamma_{b_1\alpha}^b}{\partial x^j} + \rho \Gamma_{e\beta}^b \rho \Gamma_{b_1\alpha}^e \\
& - \rho_\alpha^i \circ h \frac{\partial \rho \Gamma_{b_1\beta}^b}{\partial x^i} - \rho \Gamma_{e\alpha}^b \rho \Gamma_{b_1\beta}^e - \rho \Gamma_{b_1\gamma}^b L_{\beta\alpha}^\gamma \circ h \Bigg)
\end{aligned}
$$

and

$$
u_{b_1|\gamma} L_{\alpha\beta}^\gamma \circ h = \left(\rho_\gamma^k \circ h \frac{\partial u_{b_1}}{\partial x^k} - \rho \Gamma_{b_1\gamma}^b u_b \right) L_{\alpha\beta}^\gamma \circ h
$$

it results that

$$
\begin{aligned}
-u_b\left(\rho, h\right) \mathbb{R}_{b_1,\alpha\beta}^b - u_{b_1|\gamma} L_{\alpha\beta}^\gamma \circ h &= -L_{\alpha\beta}^\gamma \circ h \rho_\gamma^k \circ h \frac{\partial u_{b_1}}{\partial x^k} \\
&+ \left(\rho_\alpha^i \circ h \frac{\partial \rho \Gamma_{b_1\beta}^b}{\partial x^i} u_b - \rho_\beta^j \circ h \frac{\partial \rho \Gamma_{b_1\alpha}^b}{\partial x^j} u_b \right) \\
&+ \rho \Gamma_{b_1\beta}^b \rho \Gamma_{b\alpha}^a u_a - \rho \Gamma_{b_1\alpha}^b \rho \Gamma_{b\beta}^a u_a.
\end{aligned}
$$

q.e.d.

Lemma 5.2 *If* $(E, \pi, M) = (F, \nu, N)$*, then, for any*

$$
u_b s^b \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right),
$$

we have that $u_{b\ |c}$, $b, c \in \overline{1, n}$ are the *components of a tensor field of* $(0, 2)$ *type.*

Proof. Let U and U' be two vector local $(m+n)$-charts such that $U \cap U' \neq \phi$.

Since $u_{b'}(x)=M_{b'}^{b}(x)\,u_b(x)$, for any $x\in U\cap U'$, it results that

$$\begin{array}{rl}\rho_{c'}^{k'}\circ h(x)\dfrac{\partial u_{b'}(x)}{\partial x^{k'}} & =\rho_{c'}^{k'}\circ h(x)\dfrac{\partial}{\partial x^{k'}}\left(M_{b'}^{b}(x)\right)u_b(x)\\ & +M_{b'}^{b}(x)\,\rho_{c'}^{k'}\circ h(x)\dfrac{\partial u_b(x)}{\partial x^{k'}}.\end{array}\tag{1}$$

Since, for any $x\in U\cap U'$, we have

$$\begin{array}{rl}\rho\Gamma_{b'c'}^{a'}(x) & =M_a^{a'}(x)\left(\rho_c^k\circ h(x)\dfrac{\partial}{\partial x^k}\left(M_{b'}^{a}(x)\right)\right.\\ & \left.+\rho\Gamma_{bc}^{a}(x)\,M_{b'}^{b}(x)\right)M_{c'}^{c}(x),\end{array}\tag{2}$$

and

$$\begin{array}{rl}0 & =\dfrac{\partial}{\partial x^{k'}}\left(M_a^{a'}(x)\,M_{b'}^{a}(x)\right)\\ & =\dfrac{\partial}{\partial x^{k'}}\left(M_a^{a'}(x)\right)M_{b'}^{a}(x)+M_a^{a'}(x)\dfrac{\partial}{\partial x^{k'}}\left(M_{b'}^{a}(x)\right)\end{array}\tag{3}$$

it results that

$$\begin{array}{c}\rho\Gamma_{b'c'}^{a'}(x)\,u_{a'}(x)=-\rho_{c'}^{k'}\circ h(x)\dfrac{\partial}{\partial x^{k'}}\left(M_{b'}^{b}(x)\right)u_b(x)\\ +M_{b'}^{b}(x)\,\rho\Gamma_{bc}^{a}(x)\,u_a(x)\,M_{c'}^{c}(x).\end{array}\tag{4}$$

Summing the equalities (1) and (4), it results the conclusion of lemma. *q.e.d.*

Theorem 5.4 *If* $(E,\pi,M)=(F,\nu,N)$, *then, for any*

$$u_bs^b\in\Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right),$$

we shall use the notation

$$u_{b_1\ |a|b}=u_{b_1\ |ab}-\rho\Gamma_{ab}^{d}u_{b_1\ |d}\tag{5.16}$$

and we verify the formulas of Ricci type

$$u_{b_1\ |a|b}-u_{b_1\ |b|a}+(\rho,h)\,\mathbb{T}_{\ ab}^{d}u_{b_1\ |d}=-u_d\,(\rho,h)\,\mathbb{R}_{b_1\ ab}^{d}-u_{b_1\ |d}L_{ab}^{d}\circ h\tag{5.17}$$

In the particular case of Lie algebroids, $h=Id_M$, *the relations* (5.17) *become*

$$u_{b_1\ |a|b}-u_{b_1\ |b|a}+\rho\mathbb{T}_{\ ab}^{d}u_{b_1\ |d}=-u_d\rho\mathbb{R}_{b_1\ ab}^{d}-u_{b_1\ |d}L_{ab}^{d}.\tag{5.17'}$$

In the classical case, $(\rho,h)=(Id_{TM},Id_M)$, *the relations* (5.17′) *become*

$$u_{j_1\ |i|j}-u_{j_1\ |j|i}+\mathbb{T}_{\ ij}^{h}u_{j_1\ |h}=u_h\mathbb{R}_{j_1\ ij}^{h}.\tag{5.17''}$$

Theorem 5.5 *For any tensor field*

$$T_{b_1\ldots b_q}^{a_1\ldots a_p}s_{a_1}\otimes\ldots\otimes s_{a_p}\otimes s^{b_1}\otimes\ldots\otimes s^{b_q}\ ,$$

we verify the equality:

$$\begin{array}{l}T_{b_1\ldots b_q|\alpha\beta}^{a_1\ldots a_p}-T_{b_1\ldots b_q|\beta\alpha}^{a_1\ldots a_p}=T_{b_1\ldots b_q}^{aa_2\ldots a_p}(\rho,h)\,\mathbb{R}_{a\ \alpha\beta}^{a_1}+\ldots\\ +T_{b_1\ldots b_q}^{a_1\ldots a_{p-1}a}(\rho,h)\,\mathbb{R}_{a\ \alpha\beta}^{a_p}-T_{bb_2\ldots b_q}^{a_1\ldots a_p}(\rho,h)\,\mathbb{R}_{b_1\ \alpha\beta}^{b}-\ldots\\ -T_{b_1\ldots b_{q-1}b}^{a_1\ldots a_p}(\rho,h)\,\mathbb{R}_{b_q\ \alpha\beta}^{b}-T_{b_1\ldots b_q|\gamma}^{a_1\ldots a_p}L_{\alpha\beta}^{\gamma}\circ h.\end{array}\tag{5.18}$$

In the particular case of Lie algebroids, $h = Id_M$, the relations (5.18) *become*

$$(5.18') \qquad \begin{array}{l} T^{a_1...a_p}_{b_1...b_q|\alpha\beta} - T^{a_1...a_p}_{b_1...b_q|\beta\alpha} = T^{aa_2...a_p}_{b_1...b_q}\rho\mathbb{R}^{a_1}_{a\ \alpha\beta} + ... \\ +T^{a_1...a_{p-1}a}_{b_1...b_q}\rho\mathbb{R}^{a_p}_{a\ \alpha\beta} - T^{a_1...a_p}_{bb_2...b_q}\rho\mathbb{R}^{b}_{b_1\ \alpha\beta} - ... \\ -T^{a_1...a_p}_{b_1...b_{q-1}b}\rho\mathbb{R}^{b}_{b_q\ \alpha\beta} - T^{a_1...a_p}_{b_1...b_q|\gamma}L^{\gamma}_{\alpha\beta}. \end{array}$$

In the classical case, $(\rho, h) = (Id_{TM}, Id_M)$, the relations $(5.18')$ *become*

$$(5.18'') \qquad \begin{array}{l} T^{i_1...i_p}_{j_1...j_q|hk} - T^{i_1...i_p}_{j_1...j_q|kh} = T^{ii_2...i_p}_{j_1...j_q}\mathbb{R}^{i_1}_{i\ hk} + ... \\ +T^{i_1...i_{p-1}i}_{j_1...j_q}\mathbb{R}^{i_p}_{i\ hk} - T^{i_1...i_p}_{jj_2...j_q}\mathbb{R}^{j}_{j_1\ hk} - ... - T^{i_1...i_p}_{j_1...j_{q-1}j}\mathbb{R}^{j}_{j_q\ hk}. \end{array}$$

Theorem 5.6 *If $(E, \pi, M) = (F, \nu, N)$, then we obtain the following formulas of Ricci type:*

$$(5.19) \qquad \begin{array}{l} T^{a_1...a_p}_{b_1...b_q\ |b|c} - T^{a_1...a_p}_{b_1...b_q|c|b} + (\rho, h)\,\mathbb{T}^{d}_{\ bc}T^{a_1...a_p}_{b_1...b_q|d} \\ = T^{aa_2...a_p}_{b_1...b_q}(\rho, h)\,\mathbb{R}^{a_1}_{a\ bc} + ... + T^{a_1...a_{p-1}a}_{b_1...b_q}(\rho, h)\,\mathbb{R}^{a_p}_{a\ bc} \\ -T^{a_1...a_p}_{bb_2...b_q}(\rho, h)\,\mathbb{R}^{b}_{b_1\ bc} - ... - T^{a_1...a_p}_{b_1...b_{q-1}b}(\rho, h)\,\mathbb{R}^{b}_{b_q\ bc} - T^{a_1...a_p}_{b_1...b_q|d}L^{d}_{bc}\circ h. \end{array}$$

In the particular case of Lie algebroids, $h = Id_M$, the relations (5.19) *become*

$$(5.19') \qquad \begin{array}{l} T^{a_1...a_p}_{b_1...b_q\ |b|c} - T^{a_1...a_p}_{b_1...b_q|c|b} + \rho\mathbb{T}^{d}_{\ bc}T^{a_1...a_p}_{b_1...b_q|d} \\ = T^{aa_2...a_p}_{b_1...b_q}\rho\mathbb{R}^{a_1}_{a\ bc} + ... + T^{a_1...a_{p-1}a}_{b_1...b_q}\rho\mathbb{R}^{a_p}_{a\ bc} \\ -T^{a_1...a_p}_{bb_2...b_q}\rho\mathbb{R}^{b}_{b_1\ bc} - ... - T^{a_1...a_p}_{b_1...b_{q-1}b}\rho\mathbb{R}^{b}_{b_q\ bc} - T^{a_1...a_p}_{b_1...b_q|d}L^{d}_{bc}. \end{array}$$

In the classical case, $(\rho, h) = (Id_{TM}, Id_M)$, the relations $(5.19')$ *become*

$$(5.19'') \qquad \begin{array}{l} T^{i_1...i_p}_{j_1...j_q\ |h|k} - T^{i_1...i_p}_{j_1...j_q|k|h} + \mathbb{T}^{m}_{\ hk}T^{i_1...i_p}_{j_1...j_q|m} \\ = T^{ii_2...i_p}_{j_1...j_q}\mathbb{R}^{i_1}_{i\ hk} + ... + T^{i_1...i_{p-1}i}_{j_1...j_q}\mathbb{R}^{i_p}_{i\ hk} - T^{i_1...i_p}_{jj_2...j_q}\mathbb{R}^{j}_{j_1\ hk} - ... - T^{i_1...i_p}_{j_1...j_{q-1}j}\mathbb{R}^{j}_{j_q\ hk}. \end{array}$$

We observe that if the structure functions of generalized Lie algebroid

$$\left((F, \nu, M), [,]_{F,h}, (\rho, Id_M)\right),$$

the (ρ, h)-torsion associated to linear ρ-connection $\rho\Gamma$ and the (ρ, h)-curvature associated to linear ρ-connection $\rho\Gamma$ are null, then we have the equality:

$$(5.20) \qquad T^{a_1...a_p}_{b_1...b_q|b|c} = T^{a_1...a_p}_{b_1...b_q|c|b},$$

which generalizes the Schwartz equality.

Theorem 5.7 *If (E, π, M)=(F, ν, N), then the following relations hold good*

$$(\tilde{B}_1) \qquad \sum_{cyclic(U_1,U_2,U_3)} \left\{ \left(\rho\ddot{D}_{U_1}(\rho, h)\,\mathbb{T}\right)(U_2, U_3) - (\rho, h)\,\mathbb{R}(U_1, U_2)\,U_3 \right.$$
$$\left. + (\rho, h)\,\mathbb{T}\left((\rho, h)\ \ \mathbb{T}(U_1, U_2), U_3\right)\right\} = 0,$$

and

$(\tilde{B}_2)$

$$\sum_{cyclic(U_1,U_2,U_3,U)} \left\{ \left(\rho\ddot{D}_{U_1}(\rho, h)\,\mathbb{R}\right)(U_2, U_3)\,U + (\rho, h)\,\mathbb{R}\left((\rho, h)\,\mathbb{T}(U_1, U_2), U_3\right)U \right\} = 0.$$

respectively. This identities will be called the first respectively the second identity of Bianchi type.

In the particular case of Lie algebroids, $h = Id_M$, the identities $(\tilde{B}_1)$ and $(\tilde{B}_2)$ become

$$(\tilde{B}_1') \qquad \sum_{cyclic(u_1,u_2,u_3)} \left\{ \left(\rho D_{u_1} \rho \mathbb{T}\right)(u_2,u_3) - \rho \mathbb{R}(u_1,u_2)\, u_3 + \rho \mathbb{T}\left(\rho\ \mathbb{T}(u_1,u_2), u_3\right)\right\} = 0,$$

$$(\tilde{B}_2') \qquad \sum_{cyclic(u_1,u_2,u_3,u)} \left\{ \left(\rho D_{u_1} \rho \mathbb{R}\right)(u_2,u_3)\, u + \rho \mathbb{R}\left(\rho \mathbb{T}(u_1,u_2), u_3\right) u\right\} = 0.$$

In the classical case, $(\rho,h) = (Id_{TM}, Id_M)$, the identities $(\tilde{B}_1')$ and $(\tilde{B}_2')$ become

$$(\tilde{B}_1'') \qquad \sum_{cyclic(X_1,X_2,X_3)} \left\{ \left(D_{X_1} \mathbb{T}\right)(X_2,X_3) - \mathbb{R}(X_1,X_2)\, X_3 + \mathbb{T}\left(\ \mathbb{T}(X_1,X_2), X_3\right)\right\} = 0,$$

$$(\tilde{B}_2'') \qquad \sum_{cyclic(X_1,X_2,X_3,X)} \left\{ \left(D_{X_1} \mathbb{R}\right)(X_2,X_3)\, X + \mathbb{R}\left(\mathbb{T}(X_1,X_2), X_3\right) X\right\} = 0.$$

Proof: Using the equality

$$\begin{array}{c} \left(\rho \ddot{D}_{U_1}(\rho,h)\,\mathbb{T}\right)(U_2,U_3) = \rho \ddot{D}_{U_1}\left((\rho,h)\,\mathbb{T}(U_2,U_3)\right) \\ -(\rho,h)\,\mathbb{T}\left(\rho \ddot{D}_{U_1} U_2, U_3\right) - (\rho,h)\,\mathbb{T}\left(U_2, \rho \ddot{D}_{U_1} U_3\right) \end{array}$$

and the Jacobi identity we obtain the first identity of Bianchi type.

Using the equality

$$\begin{array}{l} \left(\rho \ddot{D}_{U_1}(\rho,h)\,\mathbb{R}\right)(U_2,U_3)\, U = \rho \ddot{D}_{U_1}\left((\rho,h)\,\mathbb{R}(U_2,U_3)\, U\right) \\ -(\rho,h)\,\mathbb{R}\left(\rho \ddot{D}_{U_1} U_2, U_3\right) U - (\rho,h)\,\mathbb{R}\left(U_2, \rho \ddot{D}_{U_1} U_3\right) U - (\rho,h)\,\mathbb{R}(U_2,U_3)\, \rho \ddot{D}_{U_1} U \end{array}$$

and the Jacobi identity we obtain the second identity of Bianchi type. *q.e.d.*

Remark 5.1 On components, the identities of Bianchi type become:

$$\begin{array}{r} \sum\limits_{cyclic(a_1,a_2,a_3)} \left\{ (\rho,h)\,\mathbb{T}^b{}_{a_2a_3|a_1} + (\rho,h)\,\mathbb{T}^b{}_{ga_3} \cdot (\rho,h)\,\mathbb{T}^g{}_{a_1a_2}\right\} \\ = \sum\limits_{cyclic(a_1,a_2,a_3)} (\rho,h)\,\mathbb{R}^a_{a_3\ a_1a_2} \end{array}$$

and

$$\sum_{cyclic(a,a_1,a_2,a_3)} \left\{ (\rho,h)\,\mathbb{R}^b_{a\ a_2a_3|a_1} + (\rho,h)\,\mathbb{R}^b_{a\ ga_3} \cdot (\rho,h)\,\mathbb{T}^g{}_{a_2a_1}\right\} = 0.$$

If the (ρ,h)-torsion is null, then the identities of Bianchi type become:

$$\sum_{cyclic(a_1\ a_2,a_3)} (\rho,h)\,\mathbb{R}^b_{a_3\ a_1a_2} = 0$$

and

$$\sum_{cyclic(a,a_1,a_2,a_3)} (\rho,h)\,\mathbb{R}^b_{a\ a_2a_3|_{a_1}} = 0.$$

6 (Pseudo)metrizable vector bundles. Formulas of Levi-Civita type

We will apply our theory for the diagram:

$$\begin{array}{ccc} E & & \left(F, [,]_{F,h}, (\rho, Id_N)\right) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}, \tag{6.1}$$

where $(E, \pi, M) \in |\mathbf{B}^{\mathbf{v}}|$ and $\left((F, \nu, N), [,]_{F,h}, (\rho, Id_N)\right)$ is a generalized Lie algebroid.

Definition 6.1 We will say that the *vector bundle* (E, π, M) *is endowed with a pseudometrical structure* if it exists $g = g_{ab}s^a \otimes s^b \in \mathcal{T}_2^0(E, \pi, M)$ such that for each $x \in M$, the matrix $\|g_{ab}(x)\|$ is nondegenerate and symmetric.

Moreover, if for each $x \in M$ the matrix $\|g_{ab}(x)\|$ has constant signature, then we will say that *the vector bundle* (E, π, M) *is endowed with a metrical structure.*

If $g = g_{ab}s^a \otimes s^b \in \mathcal{T}_2^0(E, \pi, M)$ is a (pseudo) metrical structure, then, for any $a, b \in \overline{1, r}$ and for any vector local $(m+r)$-chart (U, s_U) of (E, π, M), we consider the real functions

$$U \xrightarrow{\tilde{g}^{ba}} \mathbb{R}$$

such that $\left\|\tilde{g}^{ba}(x)\right\| = \|g_{ab}(x)\|^{-1}$, for any $\forall x \in U$.

Definition 6.2 We admit that (E, π, M) is a vector bundle endowed with a (pseudo)metrical structure g and with a linear ρ-connection $\rho\Gamma$.

We will say that the *linear ρ-connection $\rho\Gamma$ is compatible with the (pseudo)metrical structure g* if

$$\rho D_z g = 0,\ \forall z \in \Gamma(F, \nu, N). \tag{6.2}$$

Definition 6.3 We will say that the vector bundle (E, π, M) is *ρ-(pseudo)metrizable,* if it exists a (pseudo)metrical structure $g \in \mathcal{T}_2^0(E, \pi, M)$ and a linear ρ-connection $\rho\Gamma$ for (E, π, M) compatible with g. The id_{TM}-(pseudo)metrizable vector bundles will be called *(pseudo)metrizable vector bundles.*

In particular, if (TM, τ_M, M) is a (pseudo)metrizable vector bundle, then we will say that (TM, τ_M, M) *is a* (pseudo)Riemannian space*,* and the manifold M will be called *(pseudo)Riemannian manifold.*

The linear connection of a (pseudo)Riemannian space will be called *(pseudo)Riemannian linear connection.*

Theorem 6.1 *If* $(E, \pi, M) = (F, \nu, N)$ *and* $g \in \mathcal{T}_2^0(h^*E, h^*\pi, M)$ *is a (pseudo)metrical structure, then the local real functions*

$$\begin{aligned} \rho\Gamma_{bc}^a \ &= \frac{1}{2}\tilde{g}^{ad}\left(\rho_c^k \circ h\frac{\partial g_{bd}}{\partial x^k} + \rho_b^j \circ h\frac{\partial g_{dc}}{\partial x^j} - \rho_d^l \circ h\frac{\partial g_{bc}}{\partial x^l}\right. \\ &\left. - \left(L_{bc}^e \circ h\right) g_{ed} - \left(L_{bd}^e \circ h\right) g_{ec} + \left(L_{dc}^e \circ h\right) g_{eb}\right), \end{aligned} \tag{6.3}$$

are the components of a linear ρ-connection $\rho\Gamma$ for the vector bundle $(h^*E, h^*\pi, M)$ such that $(\rho, h)\mathbb{T} = 0$ and the vector bundle $(h^*E, h^*\pi, M)$ becomes ρ-(pseudo)metrizable. This linear ρ-connection $\rho\Gamma$ will be called the *linear ρ-connection of Levi-Civita type.*

In the particular case of Lie algebroids, $h = Id_M$, *the relations* (6.3) *become*

$$(6.3') \qquad \rho\Gamma^a_{bc} = \frac{1}{2}\tilde{g}^{ad}\left(\rho^k_c\frac{\partial g_{bd}}{\partial x^k} + \rho^j_b\frac{\partial g_{dc}}{\partial x^j} - \rho^l_d\frac{\partial g_{bc}}{\partial x^l} - L^e_{bc}g_{ed} - L^e_{bd}g_{ec} + L^e_{dc}g_{eb}\right).$$

In the classical case, $(\rho, h) = (Id_{TM}, Id_M)$, *the relations* (6.3′) *become*

$$(6.3'') \qquad \Gamma^i_{jk} = \frac{1}{2}\tilde{g}^{ih}\left(\frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{hk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h}\right).$$

Proof. Since

$$\begin{array}{rl}\left(\rho\ddot{D}_U g\right) V\otimes Z & = \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)\left((g(V\otimes Z)) - g\left(\left(\rho\ddot{D}_U V\right)\otimes Z\right)\right. \\ & \left. -g\left(V\otimes\left(\rho\ddot{D}_U Z\right)\right)\right),\ \forall U,V,Z\in\Gamma(h^*E, h^*\pi, M).\end{array}$$

it results that, for any $U,V,Z\in\Gamma(h^*E,h^*\pi,M)$, we obtain the equalities:

$$\begin{array}{ll}(1) & \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)(g(V\otimes Z)) = g\left(\left(\rho\ddot{D}_U V\right)\otimes Z\right) + g\left(V\otimes\left(\rho\ddot{D}_U Z\right)\right),\\ (2) & \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(Z)(g(U\otimes V)) = g\left(\left(\rho\ddot{D}_Z U\right)\otimes V\right) + g\left(U\otimes\left(\rho\ddot{D}_Z V\right)\right),\\ (3) & \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(V)(g(Z\otimes U)) = g\left(\left(\rho\ddot{D}_V Z\right)\otimes U\right) + g\left(Z\otimes\left(\rho\ddot{D}_V U\right)\right).\end{array}$$

We observe that $(1)+(3)-(2)$ is equivalent with the equality:

$$\begin{array}{l} g\left(\left(\rho\ddot{D}_U V + \rho\ddot{D}_V U\right)\otimes Z\right) + g\left(\left(\rho\ddot{D}_V Z - \rho\ddot{D}_Z V\right)\otimes U\right)\\ \quad +g\left(\left(\rho\ddot{D}_U Z - \rho\ddot{D}_Z U\right)\otimes V\right) = \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)(g(V\otimes Z))\\ \quad +\Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(V)(g(Z\otimes U)) - \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(Z)(g(U\otimes V)).\end{array}$$

Using the condition $(\rho, h)\mathbb{T} = 0$, which is equivalent with the equality

$$\rho\ddot{D}_U V - \rho\ddot{D}_V U - [U,V]_{h^*E} = 0,$$

we obtain the equality

$$\begin{array}{l} 2g\left(\left(\rho\ddot{D}_U V\right)\otimes Z\right) + g\left([V,U]_{h^*E}\otimes Z\right) + g\left([V,Z]_{h^*E}\otimes U\right) + g\left([U,Z]_{h^*E}\otimes V\right)\\ = \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)(g(V\otimes Z)) + \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(V)(g(Z\otimes U))\\ -\Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(Z)(g(U\otimes V)),\ \forall U,V,Z\in\Gamma(h^*E,h^*\pi,M).\end{array}$$

This equality is equivalent with the following equality:

$$\begin{array}{rl} 2g\left(\left(\rho\ddot{D}_U V\right)\otimes Z\right) & = \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)\cdot(g(V\otimes Z))\\ & +\Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(V)(g(Z\otimes U)) - \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(Z)(g(U\otimes V))\\ & +g\left([U,V]_{h^*E}\otimes Z\right) - g\left([V,Z]_{h^*E}\otimes U\right) + g\left([Z,U]_{h^*E}\otimes V\right)\end{array}$$

for any $U,V,Z\in\Gamma\left(h^{\ast}E,h^{\ast}\pi,M\right).$

If $U=S_{c},V=S_{b}$ and $Z=S_{d}$, then we obtain the equality:

$$\begin{array}{ll}2g\left(\left(\rho\Gamma_{bc}^{a}S_{a}\right)\otimes S_{d}\right) & =\rho_{c}^{k}\circ h\frac{\partial g(S_{b}\otimes S_{d})}{\partial xk}+\ \rho_{b}^{j}\circ h\frac{\partial g(S_{d}\otimes S_{c})}{\partial x^{j}}-\rho_{d}^{l}\circ h\frac{\partial g(S_{b}\otimes S_{c})}{\partial x^{l}}\\ & +g\left(\left(L_{cb}^{e}\circ h\right)S_{e}\otimes S_{d}\right)-g\left(\left(L_{bd}^{e}\circ h\right)S_{e}\otimes S_{c}\right)+g\left(\left(L_{dc}^{e}\circ h\right)S_{e}\otimes S_{b}\right),\end{array}$$

which is equivalent with:

$$\begin{array}{ll}2g_{da}\rho\Gamma_{bc}^{a} & =\rho_{c}^{k}\circ h\frac{\partial g_{bd}}{\partial xk}+\ \rho_{b}^{j}\circ h\frac{\partial g_{dc}}{\partial x^{j}}-\rho_{d}^{l}\circ h\frac{\partial g_{bc}}{\partial x^{l}}\\ & -\left(L_{bc}^{e}\circ h\right)g_{ed}-\left(L_{bd}^{e}\circ h\right)g_{ec}+\left(L_{dc}^{e}\circ h\right)g_{eb}\end{array}$$

Finally, we obtain:

$$\begin{array}{ll}\rho\Gamma_{bc}^{a} & =\frac{1}{2}\tilde{g}^{ad}\Big(\rho_{c}^{k}\circ h\frac{\partial g_{bd}}{\partial xk}+\ \rho_{b}^{j}\circ h\frac{\partial g_{dc}}{\partial x^{j}}-\rho_{d}^{l}\circ h\frac{\partial g_{bc}}{\partial x^{l}}\\ & \quad-\left(L_{bc}^{e}\circ h\right)g_{ed}-\left(L_{bd}^{e}\circ h\right)g_{ec}+\left(L_{dc}^{e}\circ h\right)g_{eb}\Big),\end{array}$$

where $\left\|\tilde{g}^{ad}\left(x\right)\right\|=\left\|g_{da}\left(x\right)\right\|^{-1}$, for any $x\in M.$ *q.e.d.*

Theorem 6.2. *If $\left(E,\pi,M\right)=\left(F,\nu,N\right)$, $g\in\mathcal{T}_{2}^{0}\left(h^{\ast}E,h^{\ast}\pi,M\right)$ is a pseudo(metrical) structure and $\mathbb{T}\in\mathcal{T}_{2}^{1}\left(h^{\ast}E,h^{\ast}\pi,M\right)$ such that its components are skew symmetric in the lover indices, then the local real functions*

$$\overset{\circ}{\rho\Gamma}{}_{bc}^{a}=\rho\Gamma_{bc}^{a}+\frac{1}{2}\tilde{g}^{ad}\left(g_{de}\mathbb{T}_{bc}^{e}-g_{be}\mathbb{T}_{dc}^{e}+g_{ec}\mathbb{T}_{bd}^{e}\right), \tag{6.4}$$

are the components of a linear ρ-connection compatible with the (pseudo) metrical structure g, where $\rho\Gamma_{bc}^{a}$ are the components of linear ρ-connection of Levi-Civita type (6.3). *Therefore, the vector bundle $\left(h^{\ast}E,h^{\ast}\pi,M\right)$ becomes ρ-(pseudo)metrizable and the tensor field $\mathbb{T}$ is the $\left(\rho,h\right)$-torsion tensor field.*

In the particular case of Lie algebroids, $h=Id_{M}$, $g\in\mathcal{T}_{2}^{0}\left(E,\pi,M\right)$ is a pseudo(metrical) structure and $T\in\mathcal{T}_{2}^{1}\left(E,\pi,M\right)$ such that its components are skew symmetric in the lover indices, then the local real functions

$$\overset{\circ}{\rho\Gamma}{}_{bc}^{a}=\rho\Gamma_{bc}^{a}+\frac{1}{2}\tilde{g}^{ad}\left(g_{de}\mathbb{T}_{bc}^{e}-g_{be}\mathbb{T}_{dc}^{e}+g_{ec}\mathbb{T}_{bd}^{e}\right), \tag{6.4$'$}$$

are the components of a linear ρ-connection compatible with the (pseudo)metrical structure g, where $\rho\Gamma_{bc}^{a}$ are the components of linear ρ-connection of Levi-Civita type (6.3$'$).

In the classical case, $\left(\rho,h\right)=\left(Id_{TM},Id_{M}\right)$, $g\in\mathcal{T}_{2}^{0}\left(TM,\tau_{M},M\right)$ is a pseudo(metrical) structure and $T\in\mathcal{T}_{2}^{1}\left(TM,\tau_{M},M\right)$ such that its components are skew symmetric in the lover indices, then the local real functions

$$\overset{\circ}{\Gamma}{}_{jk}^{i}=\Gamma_{jk}^{i}+\frac{1}{2}\tilde{g}^{ih}\left(g_{he}\mathbb{T}_{jk}^{e}-g_{je}\mathbb{T}_{hk}^{e}+g_{ek}\mathbb{T}_{jh}^{e}\right), \tag{6.4$''$}$$

are the components of a linear connection compatible with the (pseudo)metrical structure g, where Γ_{jk}^{i} are the components of linear connection of Levi-Civita type (6.3$''$).

Theorem 6.3 *If $\left(E,\pi,M\right)=\left(F,\nu,M\right)$, $g\in\mathcal{T}_{2}^{0}\left(h^{\ast}E,h^{\ast}\pi,M\right)$ is a pseudo (metrical) structure and $\rho\overset{\circ}{\Gamma}$ is the linear ρ-connection* (6.4) *for the vector bundle $\left(h^{\ast}E,h^{\ast}\pi,M\right)$, then the local real functions*

$$\rho\tilde{\Gamma}_{b\alpha}^{a}=\rho\overset{\circ}{\Gamma}{}_{b\alpha}^{a}+\tfrac{1}{2}\tilde{g}^{ac}g_{\overset{\circ}{cb|\alpha}} \tag{6.5}$$

are the components of a linear ρ-connection such that the vector bundle $(h^*E, h^*\pi, M)$ becomes ρ-(pseudo)metrizable.

In the particular case of Lie algebroids, $h = Id_M$, $g \in \mathcal{T}_2^0(E, \pi, M)$ is a pseudo(metrical) structure and $\rho\overset{\circ}{\Gamma}$ is the linear ρ-connection $(6.4')$ *for the vector bundle (E, π, M), then the local real functions*

$$\rho\tilde{\Gamma}^a_{b\alpha} = \rho\overset{\circ}{\Gamma}^a_{b\alpha} + \frac{1}{2}\tilde{g}^{ac} g_{\underset{\circ}{cb|\alpha}} \tag{6.5'}$$

are the components of a linear ρ-connection such that the vector bundle (E, π, M) becomes ρ-(pseudo)metrizable.

In the classical case, $(\rho, h) = (Id_{TM}, Id_M)$, $g \in \mathcal{T}_2^0(TM, \tau_M, M)$ is a pseudo(metrical) structure and $\rho\overset{\circ}{\Gamma}$ is the linear ρ-connection $(6.4')$ *for the vector bundle (TM, τ_M, M), then the local real functions*

$$\tilde{\Gamma}^i_{jk} = \overset{\circ}{\Gamma}^i_{jk} + \frac{1}{2}\tilde{g}^{ih} g_{\underset{\circ}{hj|k}} \tag{6.5''}$$

are the components of a linear connection such that the vector bundle (TM, τ_M, M) becomes (pseudo)metrizable.

Theorem 6.4 *If $g \in \mathcal{T}_2^0(h^*E, h^*\pi, M)$ is a pseudo (metrical) structure, $\rho\tilde{\Gamma}$ is the linear ρ-connection* (6.5) *for the vector bundle $(h^*E, h^*\pi, M)$, $T = T^d_{c\alpha} S_d \otimes S^c \otimes t^\alpha$, and $O^{ca}_{bd} = \frac{1}{2}\delta^c_b \delta^a_d - g_{bd}\tilde{g}^{ca}$ is the Obata operator, then the local real functions*

$$\rho\hat{\Gamma}^a_{b\alpha} = \rho\tilde{\Gamma}^a_{b\alpha} + \frac{1}{2} O^{ca}_{bd} T^d_{c\alpha}, \tag{6.6}$$

are the components of a linear ρ-connection such that the vector bundle $(h^*E, h^*\pi, M)$ becomes ρ-(pseudo)metrizable.

In the particular case of Lie algebroids, $h = Id_M$, $g \in \mathcal{T}_2^0(E, \pi, M)$ is a pseudo (metrical) structure, $\rho\tilde{\Gamma}$ is the linear ρ-connection $(6.5')$ *for the vector bundle (E, π, M), $T = T^d_{c\alpha} s_d \otimes s^c \otimes t^\alpha$ and and $O^{ca}_{bd} = \frac{1}{2}\delta^c_b \delta^a_d - g_{bd}\tilde{g}^{ca}$ is the Obata operator, then the local real functions*

$$\rho\hat{\Gamma}^a_{b\alpha} = \rho\tilde{\Gamma}^a_{b\alpha} + \frac{1}{2} O^{ca}_{bd} T^d_{c\alpha}, \tag{6.6'}$$

are the components of a linear ρ-connection such that the vector bundle (E, π, M) becomes ρ-(pseudo)metrizable.

In the classical case, $(\rho, h) = (Id_{TM}, Id_M)$, $g \in \mathcal{T}_2^0(TM, \tau_M, M)$ is a pseudo(metrical) structure, $\tilde{\Gamma}$ is the linear connection $(6.5'')$ *for the vector bundle (TM, τ_M, M), $T = T^l_{hk}\partial_l \otimes dx^h \otimes dx^k$ and and $O^{hi}_{jl} = \frac{1}{2}\delta^h_j \delta^i_l - g_{jl}\tilde{g}^{hi}$ is the Obata operator, then the local real functions*

$$\hat{\Gamma}^i_{jk} = \tilde{\Gamma}^i_{jk} + \frac{1}{2} O^{hi}_{jl} T^l_{hk}, \tag{6.6'}$$

are the components of a linear connection such that the vector bundle (TM, τ_M, M) becomes (pseudo)metrizable.

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